Derivatives of Inverse Functions
Derivatives of Inverse Functions

• Find the derivative of an inverse function, algebraically.
• Find the derivative of an inverse trigonometric function.
• Find the derivative of an inverse function given only points.

Language Objectives:
• Define inverse function.
• Explain the algorithm we can use to find the derivative of an inverse trigonometric function.
Finding the Derivative Algebraically

• When dealing with inverses, we are just switching the x and y. So, we can find the inverse by switching them and then use implicit differentiation (or just solve for y then derive) to get the derivative.
Example 1

• Let \( y = x^2 \), find the derivative of the inverse of \( y \).
Example 1

• Let $y = x^2$, find the derivative of the inverse of $y$.

$$\frac{dy}{dx} = \frac{1}{2y}$$
Example 2

• Let $y = x^2$, find the derivative of the inverse at $x = 4$. 
Example 2

• Let \( y = x^2 \), find the derivative of the inverse at \( x = 4 \).

• \( x = 4 \rightarrow y = 2 \)

\[ \frac{dy}{dx} = \frac{1}{4} \]
Example 3

• Find the derivative of the inverse function of \( f(x) = x^3 - 4x^2 + 7x - 1 \) at \( x = 1 \).
Example 3

• Find the derivative of the inverse function of $f(x) = x^3 - 4x^2 + 7x - 1$ at $x = 1$.

$x = 1 \rightarrow y = 0.349$

$$\frac{dy}{dx} = \frac{1}{3y^2 - 8y + 7} = 0.219$$
Example 4

• Find the derivative of the inverse function of \( f(x) = e^x + \ln x \) at \( x = 3 \)
Example 4

• Find the derivative of the inverse function of \( f(x) = e^x + \ln x \) at \( x = 3 \)

\[
\frac{dy}{dx} = \frac{1}{e^y + \frac{1}{y}} = 0.259
\]

\( x = 3 \rightarrow y = 1.074 \)
A Pattern

• $y = x^2 \rightarrow \frac{dy}{dx} = \frac{1}{2y}$

• $y = x^3 - 4x^2 + 7x - 1 \rightarrow \frac{dy}{dx} = \frac{1}{3y^2 - 8y + 7}$

• $y = e^x + \ln x \rightarrow \frac{dy}{dx} = \frac{1}{ey + \frac{1}{y}}$
The Pattern

\[ \frac{d}{dx} f^{-1}(y) = \frac{1}{f'(x)} \]
Using the Pattern

• Consider a function $f(x)$ where $f(3) = 2$ and $f'(3) = -5$.

• What is $\frac{d}{dx} f^{-1}(2)$?
Using the Pattern

• Consider a function \( f(x) \) where \( f(3) = 2 \) and \( f'(3) = -5 \).

\[
\frac{d}{dx} f^{-1}(2) = \frac{1}{f'(3)} = \frac{1}{-5}
\]
Example 5

28. Let \( f \) be a differentiable function such that \( f(3) = 15, \quad f(6) = 3, \quad f'(3) = -8, \) and \( f'(6) = -2. \) The function \( g \) is differentiable and \( g(x) = f^{-1}(x) \) for all \( x. \) What is the value of \( g'(3)? \)
Example 6

• Let $f$ be a differentiable function such that $f(-3) = 12$, $f(4) = -3$, $f'(4) = 5$, and $f'(-3) = 2$. The function $g$ is differentiable and $g(x) = f^{-1}(x) \forall x$. What is the value of $g'(-3)$?
Inverses of Trigonometric Functions

• A quick review:

\[ \sin^{-1}(x) = y \]
\[ \rightarrow x = \sin y \]

\[ \arccos(x) = y \]
\[ \rightarrow x = \cos y \]
Derivatives of Inverse Trig

• Find $\frac{d}{dx} (\sin^{-1} x)$

• Can also be written: $\sin^{-1} x = y$

• So, $\sin y = x$

• Take the derivative

• We don't want to leave it in terms of $y$, so we need a diagram.
Examples

• Find the derivative of:

• $y = \cos^{-1} x$

• $y = \tan^{-1} x$

The derivatives of the three inverse trig functions are as follows:

\[
\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}
\]

\[
\frac{d}{dx}(\cos^{-1} u) = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}
\]

\[
\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2} \frac{du}{dx}
\]
Example 6

\[ y = \sin^{-1} 4x \]
Example 7

\[ y = \tan^{-1} x^3 \]
Example 8

\[ y = \arccos(3x + 1) \]
Example 9

\[ y = \arctan(2x + 1) \]