Single Variable Calculus: Spherical Cows and Early Transcendentals

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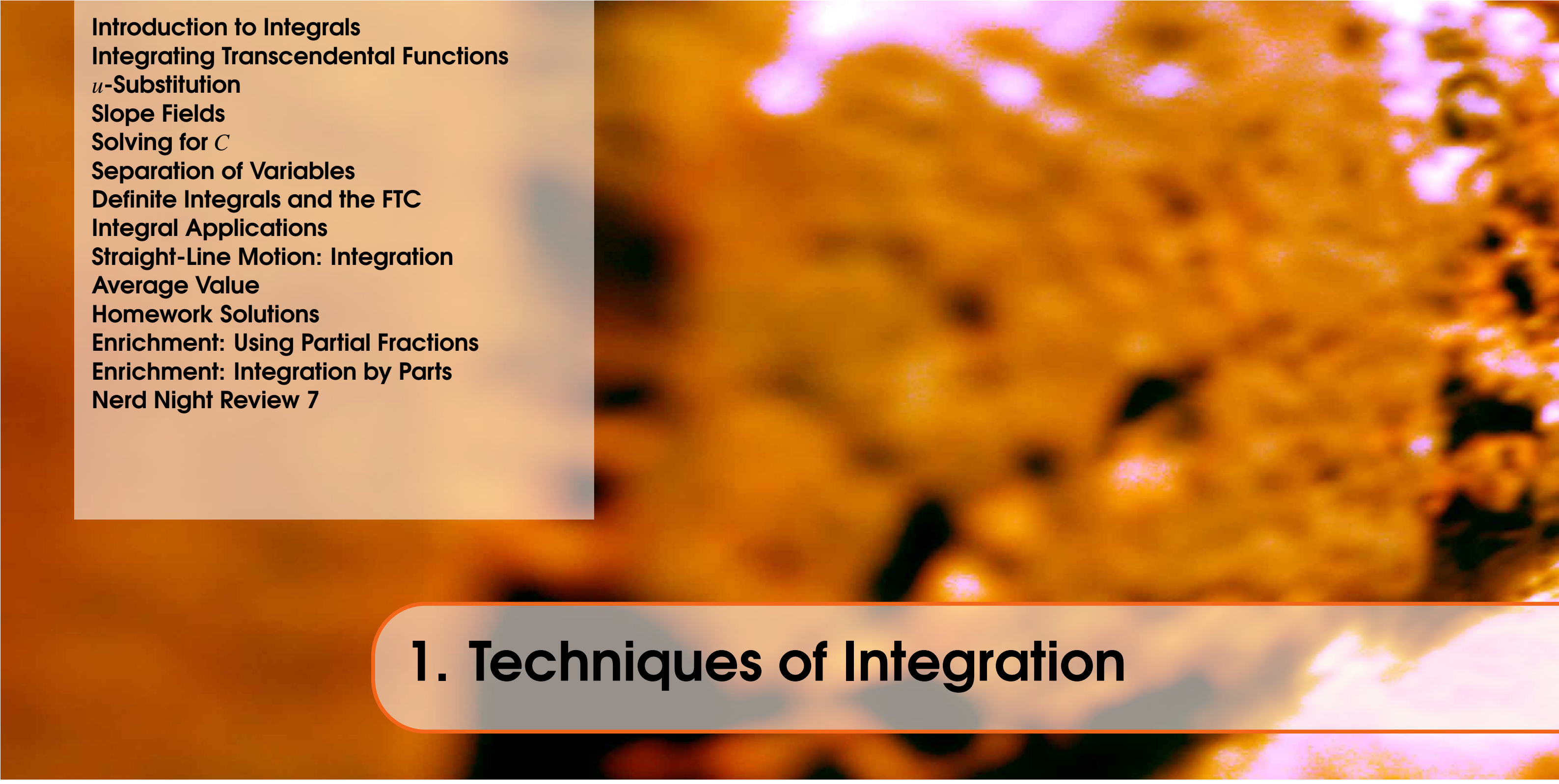


1. Techniques of Integration

Up until now, we have only learned about taking the derivative of a function. Given the position function, we can find velocity. Given the velocity function, we can find acceleration. We now need to learn to go backwards.

First Semester	Second Semester
Position	Acceleration
⇓	⇓
Velocity	Velocity
⇓	⇓
Acceleration	Position

Integrating is not quite as straight-forward as differentiating. There are, in fact, many functions which can't be integrated using basic calculus. Many a mathematician has spent his or her career finding new ways to integrate, or maybe even just working on integrating a single function. In this class, you'll learn a couple of techniques for integrating, but this is just the tip of the iceberg. As you continue taking Calculus classes, you'll continue to learn about more and more advanced techniques for taking integrals, and even still, there will always be an integral out there somewhere that you just can't do. Integration is a completely different beast than differentiation.

1.1 Introduction to Integrals

We'll begin in the same place we began with derivatives – the Power Rule. This is one of the few rules that easily converts from derivatives directly to integrals and is fairly easy to pick up.

Objectives — Introduction to Integrals.

- Use the Power Rule to find antiderivatives of functions.

The best way to learn how to do integration is to just start! But let's get do some vocabulary first.

Vocabulary — Antiderivative.
"Antiderivative" is the technically correct term to use instead of "integral." In calculus, there are two kinds of integrals – the **indefinite integral** (also called the **antiderivative**) and the **definite integral**. We will learn about definite integrals later; in the next few lessons we will focus on antiderivatives. Mathematically, they are closely related, however, the answer to an antiderivative is always an equation, whereas the answer to a definite integral is always a number.

Most people don't bother with (or aren't aware of) the distinction and simply use the term **integral**; however it is important to note that there is actually a difference.

Vocabulary — $\int$.
This is not technically a word, but this symbol is what we use to signify that we are taking the integral of something. It is the same idea as using $\frac{d}{dx}$ when we're talking about derivatives. When using $\int$, we always have a dx at the end of the function. For example: $\int (5x+2)\,\mathrm{d}x$. This is because we must specify which variable we're taking the integral with respect to (just like when we were working with derivatives). After we perform the integration, we no longer need or include the dx (you can imagine it falling off).

1.1.1 Using the Power Rule

Since it's been a while since we've done Calculus, let's have a refresher problem first.

Find $\frac{d}{dx}\left(6x^4-2x+15\right)$.

If you remember the Power Rule from last semester, you should remember that you will multiply the exponent by the corresponding coefficient and then subtract one from the exponent. When we do this, we get: $24x^3-2$

With this in mind, let's now look at an example of antidifferentiation.

Example 1.1 Find $\displaystyle\int \left(2x^9-4x^4+2x-1\right)\,\mathrm{d}x$ ■

This problem is asking us: what could you take the derivative of to get that expression? You can try to guess and check. For example: Could the answer be $19x^{10}+2x-4$? Well, hopefully it's obvious that this is not the answer. When we take the derivative of that, we would get $190x^9+2$ which is just right out.

Perhaps you can see that we could use $\frac{2}{10}x^{10}-\frac{4}{5}x^5+x^2-x$. Taking the derivative, the denominator of $\frac{2}{10}$ would cancel out with the 10 when you multiply and then the 10 would become a 9 and so on. Going through each of the terms, we see it does indeed work.

But wait! What about... $\frac{2}{10}x^{10}-\frac{4}{5}x^5+x^2-x+152$? Wouldn't that also differentiate to give us the original question? Or how about $\frac{2}{10}x^{10}-\frac{4}{5}x^5+x^2-x-4982.84$? The fact is that because the constant will always disappear when you take the derivative, technically, the integral could have any constant at the end of it. They'll all work. We'll talk about that more later.

So now, we arrive at a question: must we always guess and check? Or is there an easier way to do this? And of course, the answer is the latter.

Theorem 1.1.1 — Antidifferentiation Involving the Power Rule. When taking a derivative, we always multiply and then subtract. Since we are now doing the inverse, when we integrate, we will add and then divide. (Note that the operations changed and their order changed.)

Derivatives	**Anti-Derivatives**
Multiply $\rightsquigarrow$ Subtract	Add $\rightsquigarrow$ Divide

Let's try another example now that we know this trick.

Example 1.2 Find $\int (4x^3 - 2x^7 + 3x)\,dx$ ■

We'll use our algorithm from Theorem 1.1.1. We add one to the exponent and then divide by that number for each term. This gives us a possible answer of: $\frac{4}{4}x^4 - \frac{2}{8}x^8 + \frac{3}{2}x^2 + 672821.975493$. Remember, we could end with any constant we want because it will just fall off when we go to take the derivative anyway. Of course, you can certainly simplify this answer which would give us $x^4 - \frac{1}{4}x^8 + \frac{3}{2}x^2 + 672821.975493$.

Let's formalize this idea of the constant. Because the constant at the end could literally be anything, we want an answer that encompasses all of those possibilities. So we use the letter C (always capitalized) to represent this and we call it the "**constant of integration**." Our final answer for example 5.2 would be: $x^4 - \frac{1}{4}x^8 + \frac{3}{2}x^2 + C$. From now on, any time you're doing an anti-derivative, you MUST include the $+C$. Leaving it off will cause the problem to be wrong!

Sometimes, we'll have to simplify before we can use this rule. For example,

Example 1.3 Find $\int x^2\left(x + 3x^5\right)dx$ ■

Before we can use the Power Rule, we need to distribute. That would give us: $\int \left(x^3 + 3x^6 + 2\right)dx$

Now, we can use our trick, add then divide, which gives us our answer: $\frac{1}{4}x^4 + \frac{3}{7}x^7 + 2x + C$.

Example 1.4 Find $\int \frac{x^4 - 4x^3 + x^2 + 5}{x^2}\,dx$ ■

Just like in the last example, we can't use the Power Rule the way this is set up right now. We need to rearrange things a little. So, we're going to split this fraction into four different fractions like this: $\int \left(\frac{x^4}{x^2} - \frac{4x^3}{x^2} + \frac{x^2}{x^2} + \frac{5}{x^2}\right)dx$

Surely, we can clean this up a little bit. We can reduce each fraction by x^2 (since that's in the denominator), so we would get: $\int \left(x^2 - 4x + 1 + \frac{5}{x^2}\right)dx$.

Taking it a step further leaves us at: $\int \left(x^2 - 4x + 1 + 5x^{-2}\right)dx$.

We can now integrate which gives us the grand answer of: $\frac{1}{3}x^3 - 2x^2 - 5x^{-1} + C$.

Example 1.5 Find $\int \left(\sqrt{x} + \frac{1}{x^3}\right)dx$ ■

There is quite a bit going on here, but never fear. Recall when we were doing derivatives, we always started by rewriting everything so we didn't have fractions or radicals anymore. Let's use that same policy here: $\int\left(x^{\frac{1}{2}}+x^{-3}\right)\,dx$.

Now, we can follow the same policy we were using earlier... add and then divide by the new exponent. We get an answer of $\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{1}{2}x^{-2}+C$.

Wow! That looks awful! (No, everything in math is beautiful.) Isn't there a better way to set this up? Of course. Remember, when you're dividing by a fraction, it's the same as multiplying by the reciprocal. So, $\div\frac{3}{2}$ is the same thing as $\times\frac{2}{3}$. Knowing that, we can get a much cleaner answer: $\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{2}x^{-2}+C$.

Example 1.6 Find $\int\left(\frac{3}{4}\sqrt{x}+\frac{1}{5}x^{-4}\right)\,dx$ ■

Let's start by getting rid of that radical. We would have: $\int\left(\frac{3}{4}x^{\frac{1}{2}}+\frac{1}{5}x^{-4}\right)\,dx$. Now we can apply our same rule of adding and dividing. Remember, though, when you divide a fraction, it's the same thing as multiplying by the reciprocal. We have: $\frac{2}{3}\cdot\frac{3}{4}x^{\frac{3}{2}}+\frac{1}{-3}\cdot\frac{1}{5}x^{-3}+C$.

Cleaning this up a little gives us: $\frac{1}{2}x^{\frac{3}{2}}-\frac{1}{15}x^{-3}+C$.

1.1.2 Practice Problems

1. Find $\int 14\,dx$
2. Find $\int \pi\,dx$
3. Find $\int\left(4x^5-2x^2+5x^3-3x+1\right)\,dx$
4. Find $\int\left(\frac{4}{5}y^7+2y^{-3}-4y\right)\,dy$
5. Find $\int\left(x^{3\pi}+x^{2\pi}-4x\right)\,dx$
6. Find $\int\left(\sqrt{x}-\frac{4}{x^3}+2x\right)\,dx$
7. Find $\int\left(\frac{2}{\sqrt{x}}-\frac{4}{x^5}+5x-2\right)\,dx$
8. Find $\int x^e\,dx$
9. Find $\int (u+3)(2u-1)\,du$
10. Find $\int\left(\frac{2x^3-4x^2+3}{x^7}\right)\,dx$
11. Find $\int\left(\frac{(2x^3-4)(4x+5)}{x^6}\right)\,dx$
12. Find $\int\left(\frac{3}{2}p^{\frac{1}{3}}+2p^{\frac{1}{2}}-14\right)\,dp$
13. Find $\int\left(\frac{x^3-x^2+4x-1}{\sqrt{x}}\right)\,dx$
14. Find $\int dx$

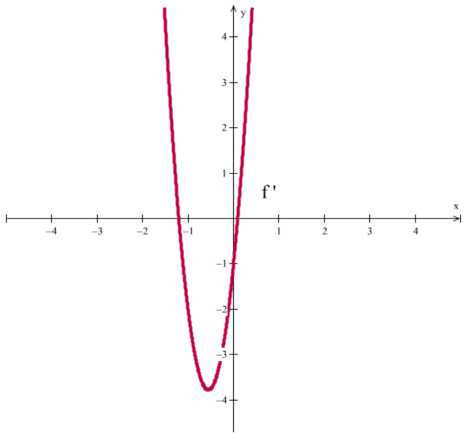
1.1.3 Practice Problems Solutions

1. $14x+C$
2. $\pi x+C$
3. $\frac{2}{3}x^6-\frac{2}{3}x^3+\frac{5}{4}x^4-\frac{3}{2}x^2+x+C$
4. $\frac{1}{10}y^8-y^{-2}-2y^2+C$
5. $\frac{x^{3\pi+1}}{3\pi+1}+\frac{x^{2\pi+1}}{2\pi+1}-2x^2+C$
6. $\frac{2}{3}x^{\frac{3}{2}}+2x^{-2}+x^2+C$
7. $4x^{\frac{1}{2}}+x^{-4}+\frac{5}{2}x^2-2x+C$
8. $\frac{x^{e+1}}{e+1}+C$
9. $\frac{2}{3}u^3+\frac{5}{2}u^2-3u+C$
10. $-\frac{2}{3}x^{-3}+x^{-4}-\frac{1}{2}x^{-6}+C$
11. $-8x^{-1}-5x^{-2}+4x^{-4}+4x^{-5}+C$
12. $\frac{9}{8}p^{\frac{4}{3}}+\frac{4}{3}p^{\frac{3}{2}}-14p+C$
13. $\frac{2}{7}x^{\frac{7}{2}}-\frac{2}{5}x^{\frac{5}{2}}+\frac{8}{3}x^{\frac{3}{2}}-2x^{\frac{1}{2}}+C$
14. $x+C$

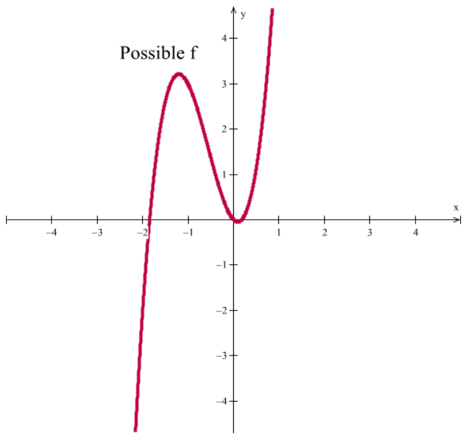
AP On the AP test, remember that they aren't terribly interested in your ability to do arithmetic and basic algebra. While those are important, they're testing your ability to do calculus. Many of the problems in this book are more complex arithmetically and algebraically than what you can expect to see in May.

1.1.4 A Word About the Constant of Integration

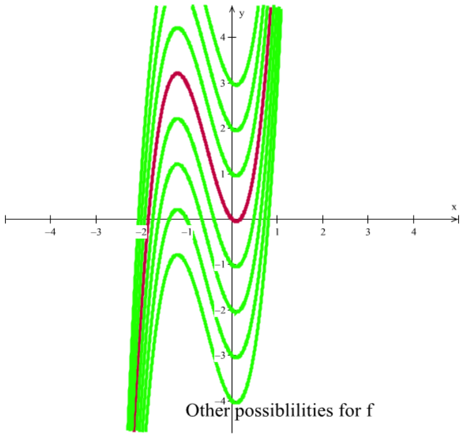
Let's take a stroll through our brains to the part where you stored information from last semester. Go specifically to the part where we studied function analysis and the graphs that went with that. Let's take the graph below as an example.



The graph shown is the derivative of some function f' and we wish to use it to graph the original function f. We note certain characteristics, for example, the $x-$intercepts would become critical values on the original function. The parts of the derivative above the $x-$axis would become increasing intervals on the original function and the parts of the derivative below the $x-$axis would translate to decreasing parts of the original function. We can even see that the minimum on this graph of the derivative would be the point of inflection for the original function. When it's all said and done, we could sketch something like:



But, recall that when doing function analysis, we discussed that there is more than one possible answer for these graphs. In fact, there are an infinite number of answers. We can translate that same graph up and down as much as we want and it would still be a perfectly valid answer. Here are some other possibilities:



So, how does this relate to the Constant of Integration? Let's look at the equations for all the functions in that last diagram. I happen to know them since I'm the one who graphed them:

$f(x) = 3x^3 + 5x^2 - x + 3$
$f(x) = 3x^3 + 5x^2 - x + 2$
$f(x) = 3x^3 + 5x^2 - x + 1$
$f(x) = 3x^3 + 5x^2 - x$
$f(x) = 3x^3 + 5x^2 - x - 1$
$f(x) = 3x^3 + 5x^2 - x - 2$
$f(x) = 3x^3 + 5x^2 - x - 3$
$f(x) = 3x^3 + 5x^2 - x - 4$

Notice that the part of each of these equations that is changing is the $y-$intercept. And notice also that when we talk about the Constant of Integration, it is actually the $y-$intercept.

The first graph above was the derivative. We wanted to get the original equation. When we go from the derivative to the original equation, we are integrating. When you integrate, you end up with a Constant of Integration which can be any constant. Since the Constant of Integration also happens to be the $y-$intercept when we substitute in different variables, we end up with a graph that moves up and down along the $y-$axis. This should provide a satisfactory explanation for why we had so many possible answers when we worked with these graphs.

There is much more for us to learn about the constant of integration. On page 65, we will see why, depending on the type of integral, the constant is not always necessary. This will be explained in much greater depth when we reach the geometric explanation on page **??**. There are even some conditions where we would be able to solve for the constant given enough information about the scenario. We'll see more on that on page 46.

1.1.5 Homework: Introduction to Integrals

Problem 1.1 Find $\int 4\,dx$.

Problem 1.2 Find $\int 2x\,dx$.

Problem 1.3 Find $\int e\,dx$.

Problem 1.4 Find $\int \left(3m^2 - 2m + 6m^3 - 4\right)\,dm$.

Problem 1.5 Find $\int \left(\frac{1}{2}x^{\frac{3}{2}} + 4x^{\frac{7}{3}}\right)\,dx$.

Problem 1.6 Find $\int x^{e\pi}\,dx$.

Problem 1.7 Find $\int (4y+2)^2\,dy$.

Problem 1.8 Find $\int \left(\frac{2x+4x^2}{\sqrt[3]{x}}\right)\,dx$.

Problem 1.9 Find $\int \left(\frac{4x^3+2x^2-1}{x^5}\right)\,dx$.

Problem 1.10 Find $\int x\,dy$.

1.2 Integrating Transcendental Functions

Now that we have a good general understanding of how antidifferentiation works, we're going to look specifically at the transcendental functions we learned about before. Specifically, we'll be looking at e^x, $\ln x$, and the trigonometric functions.

Objectives — Integration of Transcendental Functions.

- Use the derivatives of e^x and $\ln x$ and derivatives of the trigonometric functions to find antiderivatives of functions.

1.2.1 Working with e^x and $\ln x$

Let's once again refresh our memory on the derivatives. Recall that $\frac{d}{dx}e^x = e^x$ and $\frac{d}{dx}\ln x = \frac{1}{x}$. Going backwards, we can conclude:

Theorem 1.2.1 — Antiderivatives of e^x and $\ln x$. $\int \frac{1}{x}\,dx = \ln|x| + C$ and $\int e^x\,dx = e^x + C$

Note that an absolute value symbol appeared next to the natural log. This is because log functions only work for positive numbers. Therefore, when we find the anti-derivative of a function and get a log, we must make sure only positive numbers are being used.

Let's dive right in and give a problem a try.

Example 1.7 Find $\int \left(\frac{1}{3x} + 3e^x\right)\,dx$ ■

To make this a little easier to see, we can rewrite things: $\frac{1}{3}\int \frac{1}{x}\,dx + 3\int e^x\,dx$

Look closely and be sure you see where everything went. Now, using the information from Theorem 5.1.2, we can find our integral: $\frac{1}{3}\ln|x| + 3e^x + C$. This is still an indefinite integral, so we still need the constant of integration just as we did before.

Example 1.8 Find $\int \frac{4}{x}\,dx$ ■

Again, we should split this up to get: $4\int \frac{1}{x}\,dx = 4\ln|x| + C$.

Example 1.9 Find $\int 3e^{8x}\,dx$ ■

This is set up a little differently because of that extra 8 in the exponent. There's also a 3 out front, but we can rewrite that part no worries: $3\int e^{8x}\,dx$. So now, about this 8...

Let's think about this backwards. We know that the answer to this problem will be something we can differentiate to get e^{8x}. Recall that when you're working with the derivatives of e, the junk in the exponent always stays the same. So our anti-derivative should follow the same requirement. My answer will have an e^{8x} in it somewhere.

Let's try a sort of guess and check process with just this one requirement. Perhaps my answer is: $3\left(e^{8x}\right)+C$ If I check this by differentiating, I would get... $3\left(8e^{8x}\right)=24e^{8x}$. We can see this doesn't quite work. We shouldn't have a 24 in front like that.

What if we consider using a $\frac{1}{8}$ to evade this issue? Perhaps our answer could be: $3\left(\frac{1}{8}e^{8x}\right)+C=\frac{3}{8}e^{8x}+C$.

Now, we can check that we do indeed have the correct answer. Taking the derivative, of our answer, we get: $\frac{3}{8}\cdot 8e^{8x}=3e^{8x}$ and our answer checks out.

Example 1.10 Find $\int\left(\frac{3}{5}e^{6x}-\frac{1}{2x}\right)\,dx$ ■

Let's begin by rewriting this in an easier form: $\frac{3}{5}\int e^{6x}\,dx-\frac{1}{2}\int\frac{1}{x}\,dx$.

We can use the same approach to the first term as we did in the Example 1.9. The second term should be pretty obviously the natural log. We get as our answer: $\frac{3}{5}\cdot\frac{1}{6}e^{6x}-\frac{1}{2}\ln|x|+C=\frac{1}{10}e^{6x}-\frac{1}{2}\ln|x|+C$.

Example 1.11 Find $\int(4x-10)e^{x^2-5x+2}\,dx$ ■

This one looks pretty tricky, but it's easier than it looks. Let's consider a similar approach to what we used in Example 1.9.

We know the junk in the exponent has to stay the same, that's part of what makes e special. So, let's try an answer of: $e^{x^2-5x+2}+C$. If we take the derivative of that, we get: $(2x-5)\,e^{x^2-5x+2}$.

We actually almost have the answer! We only need to double the $2x-5$ and we have the right derivative. That means we can use: $2e^{x^2-5x+2}+C$ as our final answer.

1.2.2 Working with the Trigonometric Functions

Once again, we have to reach back and remember our derivatives from last semester.

$\frac{d}{dx}\sin x=\cos x$

$\frac{d}{dx}\cos x=-\sin x$

$\frac{d}{dx}\tan x=\sec^2 x$

$\frac{d}{dx}\csc x=-\csc x\cot x$

$\frac{d}{dx}\sec x=\sec x\tan x$

$\frac{d}{dx}\cot x=\csc^2 x$

Using these, we can easily go backwards and figure out our antiderivatives.

Theorem 1.2.2 — Antiderivatives of Trigonometric Functions.

$\int\cos x\,dx=\sin x+C$

$\int\sin x\,dx=-\cos x+C$

$\int\sec^2 x\,dx=\tan x+C$

$\int\csc x\cot x\,dx=-\csc x+C$

$\int\sec x\tan x\,dx=\sec x+C$

$\int\csc^2 x\,dx=-\cot x+C$

Let's try out some examples.

Example 1.12 Find $\int -7\cos x\,dx$ ■

Let's begin by rewriting this: $-7\int\cos x\,dx$.

So now, all we really need t do is figure out what I can take the derivative of to get $\cos x$ and of course, that would be $\sin x$. So, my answer is: $-7\sin x + C$.

⋆⋆⋆

One of the biggest problems students have with integrating trigonometric functions is remembering which ones change signs. The greatest advice would be to write down all six of the trigonometric derivatives before beginning the test so you can simply glance at them later on if you need them.

There's another well-known trick for $\sin x$ and $\cos x$ that we can use. Begin by writing $\sin x$. Draw a down arrow and write the derivative of $\sin x$, then draw a down arrow and write the derivative of that, etc. Keep going until you get back to the $\sin x$ you started with. You should get something like this:

$$\begin{array}{c} \sin x \\ \Downarrow \\ \cos x \\ \Downarrow \\ -\sin x \\ \Downarrow \\ -\cos x \\ \Downarrow \\ \sin x \end{array}$$

To integrate, find where you are on the chart and then go up. For example, $\int -\sin x \, dx = \cos x + C$.

Let's look at some harder examples.

Example 1.13 Find $\displaystyle\int \frac{\cos x}{\sin^2 x}\, dx$ ■

We'll need to simplify this first so we can better see what's happening. We can rewrite it as: $\displaystyle\int \frac{\cos x}{\sin x}\frac{1}{\sin x}\, dx = \int \cot x \csc x \, dx = -\csc x + C$.

So, how would you have known to break up the fraction like that if I hadn't just shown you in this example? One tip would be to find a way to make it look like one of the four expressions:

$$\sec^2 x \qquad \csc^2 x \qquad \sec x \tan x \qquad \csc x \cot x \tag{1.1}$$

If you can get it there, you're home free. If you can't, then it's probably going to be an anti-derivative that's beyond the scope of this class (so you shouldn't see anything that won't work with this tip within this class).

In case you're a little rusty on your trigonometry, these are the most important identities to be familiar with (from Pre Calculus):

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x} \qquad \csc x = \frac{1}{\sin x}\left(\csc^2 x = \frac{1}{\sin^2 x}\right) \qquad \sec x = \frac{1}{\cos x}\left(\sec^2 x = \frac{1}{\cos^2 x}\right)$$

Example 1.14 Find $\int \left(\sqrt{x} - \frac{3}{\sin^2 x}\right)\,\mathrm{d}x$ ■

Rewrite this as necessary so you can see what we have: $\int x^{\frac{1}{2}}\,\mathrm{d}x - 3\int \csc^2 x\,\mathrm{d}x$.

We can integrate the first term using the inverse of the Power Rule from the last section. That will get us to the point: $\frac{2}{3}x^{\frac{3}{2}} - 3\int \csc^2 x\,\mathrm{d}x$.

Now, to handle the last part, note that I intentionally forced it to look like one of the expressions from Equation 1.1.

Thinking back to our derivatives, we know that $\frac{d}{dx}\cot x = -\csc^2 x$. This tells us that $\int \csc^2 x\,\mathrm{d}x = -\cot x + C$.

That means our final answer must be: $\frac{2}{3}x^{\frac{3}{2}} + 3\cot x + C$.

Example 1.15 Find $\int \left(\sin^2\theta + \cos^2\theta\right)\,\mathrm{d}\theta$ ■

This is a hard question, because we don't know any derivatives or integrals that have just a $\sin^2 x$ or $\cos^2 x$. Integrating them would be a complicated process indeed and would involve methods of integration that we can't even imagine.

If we think for a moment, though, we recall something from our Pre-Calculus class and that is the Pythagorean Identity. We recall that $\sin^2 x + \cos^x = 1$.

So actually, we're just integrating: $\int 1\cdot\mathrm{d}x$ which gives us our answer of: $x + C$.

Note that normally we would simply write $\int 1\cdot\mathrm{d}x$ as just $\int \mathrm{d}x$.

Example 1.16 Find $\int \sec 3x \tan 3x\,\mathrm{d}x$ ■

You should recognize the form $\sec x \tan x$. Recall that $\frac{d}{dx}\sec x = \sec x \tan x$.

From this point, we want to use a similar approach to what we did on Example 1.9. The junk in the sec and tan stays the same when we take the derivative, it's just a matter of the coefficient on the outside. So I know that integrating, I should end up with $3x$ as the angle no matter what.

I'm going to guess that my answer is $\sec 3x + C$. If we check that by taking the derivative, we get $3\sec 3x \tan 3x$ which is off by a factor of 3.

That means we'll want to divide by 3 to make it cancel out and we get our answer of: $\frac{1}{3}\sec 3x + C$.

AP The AP test will most likely focus on $\sin x$ and $\cos x$, however, it's important to be familiar with all of the anti-derivatives for the trigonometric functions, because you never know what will come up!

1.2.3 Practice Problems

1. Find $\int \left(\frac{4}{x}+e^x\right)\,dx$.
2. Find $\int e^{9x}\,dx$.
3. Find $\int 2e^{5x}\,dx$.
4. Find $\int \left(\frac{3x^2+2x-1}{x}\right)\,dx$.
5. Find $\int \frac{1}{3x}\,dx$.
6. Find $\int e^{4+2x}\,dx$.
7. Find $\int \left(3x^2+4x-5\right)e^{x^3+2x^2-5x+4}\,dx$.
8. Find $\int (6x-21)\,e^{x^2-7x+5}\,dx$.
9. Find $\int \frac{1}{2}\sec x\tan x\,dx$.
10. Find $\int \frac{3}{\cos^2(3x)}\,dx$.
11. Find $\int 2\sin\theta\cos\theta\,d\theta$.
12. Find $\int \csc\theta\,(\sin\theta+\csc\theta)\,d\theta$.
13. Find $\int \frac{x}{2}-\sin x\,dx$.
14. Find $\int \sin 4x\,dx$.
15. Find $\int \csc^2(12x)\,dx$.
16. Find $\int \left(\frac{\sin x}{\cos^2 x}+x^3\right)\,dx$.

1.2.4 Practice Problems Solutions

1. $4\ln|x|+e^x+C$
2. $\frac{1}{9}e^{9x}+C$
3. $\frac{2}{5}e^{5x}+C$
4. $\frac{3}{2}x^2+2x-\ln|x|+C$
5. $\frac{1}{3}\ln|x|+C$
6. $\frac{1}{2}e^{4+2x}+C$
7. $e^{x^3+2x^2-5x+4}+C$
8. $3e^{x^2-7x+5}+C$
9. $\frac{1}{2}\sec x+C$
10. $\tan(3x)+C$
11. $-\frac{1}{2}\cos(2\theta)+C$
12. $\theta-\cot\theta+C$
13. $\frac{1}{4}x^2+\cos x+C$
14. $-\frac{1}{4}\cos(4x)+C$
15. $-\frac{1}{12}\cot(12x)+C$
16. $\sec x+\frac{1}{4}x^4+C$

1.2.5 Homework: Integrating Transcendental Functions

Problem 1.11 Find $\int \left(\frac{3}{2x}+e^{2x}\right)\,dx$.

Problem 1.12 Find $\int \left(4x-e^{x+4}\right)\,dx$.

Problem 1.13 Find $\int 3e^{2x}\,dx$.

Problem 1.14 Find $\int \left(\frac{4x^{\frac{1}{2}}-2x}{x^{\frac{3}{2}}}\right)\,dx$.

Problem 1.15 Find $\int \left(\frac{e^{4x}+2e^x}{e^x}\right)\,dx$.

Problem 1.16 Find $\int 9e^{3x}\,dx$.

Problem 1.17 Find $\int \left(\frac{4x^3+2x^2-1}{x^4}\right)\,dx$.

Problem 1.18 Find $\int 4\sin(3x)\,dx$.

Problem 1.19 Find $\int \sec^2(2x)\,dx$.

Problem 1.20 Find $\int \left(\frac{x}{3}-\csc x\cot x\right)\,dx$.

Problem 1.21 Find $\int \sec x\,(\cos x+\sec x)\,dx$.

Problem 1.22 Find $\int (4x+3x^2+\cos x)\, \mathrm{d}x$.

Problem 1.23 Find $\int \left(\frac{4}{\cos^2 x}+\frac{2}{x}-\frac{3}{x^2}+\cos x\right)\, \mathrm{d}x$.

Problem 1.24 Find $\int \cos x\, \mathrm{d}y$.

1.3 u-Substitution

Sometimes, antiderivatives can be difficult, if not impossible, to find. All of the examples we did involved straight-forward approaches. Today, we will look at examples that will require a little finesse. It's important to note two important things that are often confusing misconceptions for students:

1. u-Substitution won't work for all antiderivatives. Sometimes an antiderivative is too complex and requires something more advanced.
2. You don't always have to use u-substitution. Often, a problem requires only what we did in the previous section, but a student far over-complicates it and attempts to implement u-substitution unnecessarily.
3. u-substitution is one method of many. It is by no means a universal way to approach integrals. You must develop the ability to determine when it should be used and when it is not appropriate.

Objectives — u-Substitution.

- Identify the composed functions and decompose them.
- Use the method of u-substitution to find antiderivatives.

Recall the approach we took when we were looking at the Chain Rule. We considered the functions as compositions and broke them apart. For example, $\frac{\mathrm{d}}{\mathrm{d}x}\sin 4x^3$.

We see that we have an outside function of $\sin u$ and an inside function of $(4x^3)$. So, when we take the derivative, we will have two steps. One for the outside function and one for the inside function: $\frac{\mathrm{d}}{\mathrm{d}x}\sin(4x^3) = \cos(4x^3)\cdot 12x^2$.

In order to do differentiation by the Chain Rule, you had to become proficient at identifying functions that were lodged inside of other functions.

To do u-Substitution, you will have the same task. You will need to identify the functions that are inside of other functions. As a positive note, you had to sometimes identify many layers of functions when you were doing the Chain Rule. With the u-Substitution problems presented in this book, however, you will only ever be given two layers of functions.

Let's begin by looking at some examples.

1.3.1 Basic Antiderivatives

Example 1.17 Find $\int 2(2x-5)^{10}\, \mathrm{d}x$ ■

We can see in this example, that taking the derivative the way we would normally do would be very difficult. We would need to multiply $(2x-5)$ by itself 10 times which is tedious and unnecessary. This is a job for $u-$substitution.

Our general approach for these problems will be to find the inside part of the function and replace it with a dummy variable that we will call u (the choice of the letter u is simply based on convention – everyone uses u, so we will use it too).

Notice that we have one function composed of another function. Specifically, u^{10} is composed of $x-5$. So, we will use the inside function, the $x-5$ as our u:

$$u = 2x - 5.$$

Our goal will be to ultimately rewrite the entire integrand in terms of u, completely eliminating all of the x values. Therefore, we need to find a way to get rid of the $\mathrm{d}x$ (since that's the only other $x-$value in there).

Here's a trick mathematicians have figured out. If we take the derivative of that u, we would have: $\frac{du}{dx} = 2$. Notice what happens if I multiply both sides by dx. I get:

$$\mathrm{d}u = 2\,\mathrm{d}x.$$

Let's go back to the integral we started with an see how all these us help us: $\int 2(2x-5)^{10}\,\mathrm{d}x$

We can rearrange this using the commutative property: $\int (2x-5)^{10} \cdot 2\,\mathrm{d}x$

Notice now that we can swap out all of the $x-$values: the $2x-5$ will be replaced by u and the $2\,\mathrm{d}x$ will be replaced by $\mathrm{d}u$. Rewriting that all, we have:

$$\int u^{10}\,\mathrm{d}u.$$

We can now integrate much more easily than we could have before. We will just treat the u as we would an x. We get:

$$\tfrac{1}{11}u^{11} + C$$

That's technically our answer, but to be proper we should write it using the original notation. Since we defined u to be $2x-5$, we will substitute that back in for our final answer:

$$\tfrac{1}{11}(2x-5)^{11} + C$$

Example 1.18 Find $\displaystyle\int (3x+2)^7\,\mathrm{d}x$ ■

We can see here that the inside function would be $3x+2$, so we can make that our $u-$value.

$$u = 3x + 2$$

We have a problem, though, when we go to work with $\frac{\mathrm{d}u}{\mathrm{d}x}$. We would have: $\frac{\mathrm{d}u}{\mathrm{d}x} = 3$ which would give us: $\mathrm{d}u = 3\,\mathrm{d}x$. The 3 is an issue. If we try to replace that in the original equation, $\int (3x+2)^7\,\mathrm{d}x$, we wouldn't be able to make a good replacement. The $3x+2$ would be replaced by u and the dx would be replaced by du, but what about the extra 3?

Let's go back to the $\mathrm{d}u = 3\,\mathrm{d}x$. If we decide both sides by 3, then we get:

$$\frac{1}{3}\,\mathrm{d}u = \mathrm{d}x$$

We can actually replace the parts of our integrand now. Replacing everything, we would get: $\int u^7 \frac{1}{3}\,\mathrm{d}u$. Admittedly, this looks awkward because the dx was replaced with the entire $\frac{1}{3}\,\mathrm{d}u$. We can easily rearrange, though:

$$\frac{1}{3}\int u^7\,\mathrm{d}u$$

Integration of this gives us:

$$\frac{1}{3}\cdot\frac{1}{8}u^8 + C$$

We should finally replace the u with what we started with to be proper:

$$\frac{1}{24}(3x+2)^8 + C$$

Example 1.19 Find $\int \frac{2t}{(t^2+3)^9}\,\mathrm{d}t$ ■

This may seem confusing because there is a lot going on. When we encounter a problem like this, sometimes it helps to just try something and see if it works. If it doesn't work, try something different. Let's just pick something:

$$u = (t^2+3)^9$$

That gives us $\frac{\mathrm{d}u}{\mathrm{d}x} = 9(t^2+3)\cdot 2t$ and so

$$\mathrm{d}u = 9(t^2+3)\cdot 2t\,\mathrm{d}x$$

Hopefully it's obvious that this is completely not going to work. There is way too much stuff there that is not in the original function that would have nowhere to go.

Let's try again with a different $u-$value:

$$u = t^2+3$$

This would give us:

$$\mathrm{d}u = 2t\,\mathrm{d}t$$

If we rearrange the original integral a little bit: $\int \frac{2t\,\mathrm{d}t}{(t^2+3)^9}$, we can clearly see that du will replace the numerator and u will replace the part in the parentheses of the denominator. This is what we want, so now we can see we have chosen the correct u.

$$\int \frac{\mathrm{d}u}{u^9} = \int u^{-9}\,\mathrm{d}u = \frac{-1}{8}u^{-8} + C = \frac{-1}{8}(t^2+3)^{-8} + C$$

Example 1.20 Find $\int [(p-1)^5 + (p-1)^2 - (p-1)]\,\mathrm{d}p$ ■

This may look confusing, but it actually has a very simple solution. Consider:

$$u = p-1,$$

which of course gives us

$$\mathrm{d}u = 1 \cdot \mathrm{d}p.$$

Replacing everything, we would have:

$$\int (u^5 + u^2 - u)\,\mathrm{d}u$$

This is now just a polynomial, and we can easily integrate:

$$\tfrac{1}{6}u^6 + \tfrac{1}{3}u^3 - \tfrac{1}{2}u^2 + C,$$

and finally write our answer:

$$\tfrac{1}{6}(p-1)^6 + \tfrac{1}{3}(p-1)^3 - \tfrac{1}{2}(p-1)^2 + C.$$

1.3.2 Working with e^x and $\ln x$

Integrals involving $u-$substitution and e^x or $\ln x$ may seem confusing, but with more practice, it becomes apparent that it's the same process over and over again. Let's dive right in to some examples.

Example 1.21 Find $\int 2e^{2x-5}\,\mathrm{d}x$ ■

It is possible to determine the answer to this without using $u-$substitution. However, it is more likely that a person might think they could find the answer without using $u-$substitution, but get the answer wrong. It is always better to use $u-$substitution when necessary than to guess at the problem.

When working with an exponential function, the exponent will almost always be our u. Let's try setting this up and see how it works.

$$u = 2x - 5$$

This sets us up with:

$$\mathrm{d}u = 2\,\mathrm{d}x$$

If we rewrite the original integral, we can see:

$$\int e^{2x-5}\,2\,\mathrm{d}x$$

The exponent can be replaced by u and the $2\,\mathrm{d}x$ can be replaced with $\mathrm{d}u$. Replacing everything, we have:

$$\int e^u\,\mathrm{d}u,$$

which easily integrates:

$$e^u + C = e^{2x-5} + C$$

Example 1.22 Find $\int (3x-1)e^{2x^3-2x+5}\,\mathrm{d}x$ ■

This might look a little more complicated, but it's really the same as the example we just did. As mentioned earlier, setting the exponent equal to u is generally the way to go:

$$u = 2x^3 - 2x + 5$$
$$du = (6x - 2)\,dx$$

In this case, we can see that the du doesn't quite match what we have in the integral. That's no problem, though – we can easily modify du to fit our needs:

$$\tfrac{1}{2}\,du = (3x - 1)\,dx$$

We can now rewrite our integral:

$$\int e^u \cdot \tfrac{1}{2}\,du = \tfrac{1}{2}\int e^u\,du = \tfrac{1}{2}e^{2x^3-2x+5} + C$$

Example 1.23 Find $\int \frac{1}{3}e^{3x+2}\,dx$ ■

We can start by trying the same trick:

$$u = 3x + 2$$
$$du = 3\,dx$$

There is a problem, though. du calls for $3\,dx$, however, in our integral, we have $\frac{1}{3}\,dx$.

There are two things we can adjust to make this work. First, we can take the $\frac{1}{3}$ out of the integral completely and forget about it for the moment:

$$\tfrac{1}{3}\int e^{3x+2}\,dx$$

Second, we can modify our du to account for the 3:

$$\tfrac{1}{3}\,du = dx$$

Now, we're ready to plug the us back into the original expression.

$$\tfrac{1}{3}\int e^u \cdot \tfrac{1}{3}\,du = \tfrac{1}{3}\cdot\tfrac{1}{3}\int e^u\,du$$

We can now integrate for our final answer:

$$\tfrac{1}{9}e^u + C = \tfrac{1}{9}e^{3x+2} + C$$

Example 1.24 Find $\int e^{\cos x}\,dx$ ■

We begin the same as in the past, with the exponent as our u.

$$u = \cos x$$
$$du = -\sin x\,dx$$

We find that we have a problem here: there is no $\sin x$ to use from the original expression and there's also no way to modify du using simple multiplication as we've done in other problems.

In this case, we conclude that $u-$substitution does not work and we'd have to use some other integration strategy (which is beyond the level of this course).

AP On the AP test, all of the integration problems will either not need any special techniques, will require the use of $u-$substitution, or will be performed directly on your calculator. There is no need to worry about integrals that are more complex than this for now. However, it is beneficial to be aware of the added complexity you will encounter in future Math classes.

Example 1.25 Find $\int \frac{1}{3x+1}\,\mathrm{d}x$ ■

Before we can begin this problem, it is important to note that $\int \frac{1}{x}\,\mathrm{d}x = \ln x + C$.

Just as with the exponential functions, there is a means of choosing u that nearly always works. The convention is to choose the denominator to be the u.

$$u = 3x+1$$
$$\mathrm{d}u = 3\,\mathrm{d}x$$
$$\frac{1}{3}\,\mathrm{d}u = \mathrm{d}x$$

We can now replace the x parts of the original integral:

$$\frac{1}{3}\int \frac{1}{u}\,\mathrm{d}u,$$

which finally allows us to integrate for our answer:

$$\frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|3x+1| + C$$

Example 1.26 Find $\int \frac{\sin(x)}{\cos(x)+3}\,\mathrm{d}x$ ■

Even though this may look a little more difficult, we can still use the same convention – set the u equal to the denominator:

$$u = \cos(x)+3$$
$$\mathrm{d}u = -\sin(x)\,\mathrm{d}x$$
$$-\mathrm{d}u = \sin(x)\,\mathrm{d}x$$

We can rewrite the original integral slightly to get a better grasp of what we're doing:

$$\int \frac{\sin(x)\,\mathrm{d}x}{\cos(x)+3}$$

Now, replacing the $x-$values with $u-$values should be straight-forward:

$$-\int \frac{\mathrm{d}u}{u} = -\ln|u| + C = -\ln|\cos(x)+3| + C$$

Perhaps my savvy readers will note that $\cos(x)+3 > 0, x \in \mathbf{R}$ *(since the range of* $\cos(x)$ *is limited to* $-1 \le y \le 1$*) and so the use of the absolute value bars in this particular answer is unnecessary.*

Example 1.27 Find $\int \frac{1}{3x^2}\,\mathrm{d}x$ ■

Once again, let's begin by using the denominator as our u.

$$u = 3x^2$$
$$\mathrm{d}u = 6x\,\mathrm{d}x$$

This poses a problem, however, as we don't have an extra x in the numerator and no way to get one by simply multiplying numbers. When this situation arises, then we know that $u-$substitution was not the right way to approach it. Indeed, we could rewrite our original integral:

$$\frac{1}{3}\int x^{-2}\,dx,$$

and then integrate it using our normal "add and divide" approach:

$$-\frac{1}{3}x^{-1}+C=-\frac{1}{3x}+C$$

AP Sometimes, it's hard to remember with the fractions when to use natural log and when to just try integrating normally. If you are unsure, always try integrating normally. If you "add and divide" and find yourself with a zero as your exponent and a zero in the denominator, that's the indicator that it should have an $\ln x$.

1.3.3 Working with Trigonometric Functions

Working with trigonometric functions will be much the same as the other two functions. Once again, we are looking for a function inside of a function. Let's see some examples.

Example 1.28 Find $\int \sin(2x-4)\,dx$ ■

We see that the interior function would be:

$$u=2x-4$$

Using the same trick we did before, we get: $du=2\,dx$. We would like to replace only the dx, so we will need to divide both sides by 2 which gives us:

$$\frac{1}{2}\,du=dx$$

We can now replace this in the original integrand just as we did before:

$$\frac{1}{2}\int \sin u\,du$$

This leaves us with the integration process:

$$\frac{1}{2}\cdot(-\cos u)+C=-\frac{1}{2}\cos(2x-4)+C$$

Example 1.29 Find $\int (8x-8)\sec^2(2x^2-4x+5)\,dx$ ■

We follow exactly the same process as before:

$$u=2x^2-4x+5$$
$$du=(4x-4)\,dx$$
$$2\,du=(8x-8)\,dx$$

We can now substitute and antidifferentiate:

$$2\int \sec^2 u\,du$$
$$=2\tan u+C=2\tan(2x^2-4x+5)+C$$

Example 1.30 Find $\int \sec 4x \tan 4x \, dx$ ■

$$u = 4x$$
$$du = 4\,dx$$
$$\tfrac{1}{4}\,du = dx$$
$$\tfrac{1}{4}\int \sec u \tan u \, du$$

It is easy to become confused by the extra bits in the integrand, but just remember, $\frac{d}{dx}\sec x = \sec x \tan x$.

$$\tfrac{1}{4}\int \sec u \tan u \, du$$
$$= \tfrac{1}{4}\sec u + C = \tfrac{1}{4}\sec 4x + C$$

1.3.4 Monkey Wrenches

Of course, regardless of how well we master the types of problems shown in the examples above, there will always be an odd question here and there. We can look at a few here.

Example 1.31 Find $\int \left(x \sin\left(8x^2 - 2\right) + x^3 - \frac{4}{x^2}\right) dx$ ■

Notice in this problem that only part of it will need $u-$substitution. The second two terms, $x^3 - \frac{4}{x^2}$ can be integrated using the rules for polynomials. With this in mind, we should split the integral into two parts – the part that requires $u-$substitution and the part that works by simply "adding and dividing."

$$\int x\sin(8x^2-2)\,dx + \int \left(x^3 - \tfrac{4}{x^2}\right) dx$$

Since the right side is straight-forward, we can integrate that first:

$$\int x\sin(8x^2-2)\,dx + \int (x^3 - 4x^{-2})\,dx$$
$$= \int x\sin(8x^2-2)\,dx + \tfrac{1}{4}x^4 + 4x^{-1} + C$$

Now, we are prepared to integrate the first term. We will choose:

$$u = 8x^2 - 2$$
$$du = 16x\,dx$$
$$\tfrac{1}{16}\,du = x\,dx$$

When we substitute in, we should always keep everything – even the part we have already integrated:

$$= \tfrac{1}{16}\int \sin u \, du + \tfrac{1}{4}x^4 + 4x^{-1} + C$$
$$= \tfrac{1}{16}\cdot(-\cos u) + \tfrac{1}{4}x^4 + 4x^{-1} + C$$
$$= -\tfrac{1}{16}\cos u + \tfrac{1}{4}x^4 + 4x^{-1} + C$$
$$= -\tfrac{1}{16}\cos\left(8x^2-2\right) + \tfrac{1}{4}x^4 + 4x^{-1} + C$$

Example 1.32 Find $\int \sec^2 x \tan x \, dx$ ■

This is an interesting problem because it actually has more than one possible answer.

Option 1: Perhaps the most obvious way to approach this is to set $u = \tan x$ so that $\mathrm{d}u = \sec^2 x$. Substituting in, we would have:

$$\int u\,\mathrm{d}u = \frac{1}{2}u^2 + C$$

So, once we get rid of the us again, we end up with an answer: $\frac{1}{2}\tan^2 x + C$

Option 2: We could have also set $u = \sec x$ which means our $\mathrm{d}u = \sec x \tan x\,\mathrm{d}x$. Substituting in, we actually have the same as we did in option 1:

$$\int u\,\mathrm{d}u = \frac{1}{2}u^2 + C$$

However, substituting back in, we end up with the answer $\frac{1}{2}\sec^2 x + C$.

If you have trouble seeing this, consider that $\sec^2 x \tan x$ *could be rewritten as* $\sec x \sec x \tan x$.

Conclusion: Both of these answers are correct answers (find the derivative of each of them to see that you will get the original expression). There will not be any ambiguous questions like this on the AP test, however, the concept that integrals can function this way is important.

Example 1.33 Find $\displaystyle\int \sin x^2 \cos x\,\mathrm{d}x$ ■

Let's begin by setting $u = \sin x^2$ which gives us $\mathrm{d}u = 2x\cos x^2\,\mathrm{d}x$. It should be obvious that this will not work for what we have been given.

Let's try again by using $u = x^2$ and $\mathrm{d}u = 2x\,\mathrm{d}x$. This will certainly not work either as we end up without a cosine term.

What if we try $u = \cos x$ and $\mathrm{d}u = -\sin x\,\mathrm{d}x$? Again, we run into a dead end.

This is an important concept: not every problem can be solved by using $u-$substitution. To solve this problem, you would need to use another technique like integration by parts which will not be addressed here. However, the reader is encouraged to research the topic for familiarity.

Example 1.34 Find $\displaystyle\int (3x^2 + 2x)\,\mathrm{d}x$ ■

If we try to establish a u, we will find the same issues that we found in the previous example. This is because we would not need to use $u-$substitution for this particular problem. This is a simple polynomial, we could easily obtain: $x^3 + x^2 + C$.

1.3.5 Practice Problems

Evaluate each of the following. If it cannot be performed using skills you have already learned, write that.

1. $6\int (x\cos(3x^2 + 2)\,\mathrm{d}x$
2. $\int \sec^2(8x)\,\mathrm{d}x$
3. $\int \cos(\ln x)\,\mathrm{d}x$
4. $\int (e^x \sin(e^x) + e^{2x} - e^{3x})\,\mathrm{d}x$
5. $\int (6x-3)^2\,\mathrm{d}x$
6. $\int (6x^2-3)^2\,\mathrm{d}x$

7. $\int \frac{2x}{3x^2-5}\,dx$
8. $\int \frac{2x}{(3x^2-5)^7\,dx}$
9. $\int \{(x^2+5)^5-(x^2+5)^3+(x^2+5)\}2x\,dx$
10. $\int \frac{2t}{4t^2-3}\,dt$
11. $\int 4ve^{v^2-8}\,dv$
12. $\int \sec(p+1)\tan(p+1)\,dp$
13. $\int \frac{\sec^2(3y-1)}{\tan(3y-1)}\,dy$
14. $\int (2h^3-1)\sin(h^4-2h)\,dh$
15. $\int \sec^2(r)e^{\tan(r)}\,dr$
16. $\int \left(\frac{2x}{x^2-4}+3x^3-2\right)\,dx$

1.3.6 Practice Problems Solutions

1. $\sin(3x^2+2)+C$
2. $\frac{1}{8}\tan(8x)+C$
3. Cannot be evaluated with what we know.
4. $-\cos(e^x)+\frac{1}{2}e^{2x}-\frac{1}{3}e^{3x}+C$
5. $\frac{1}{12}((6x-3)^3)+C$
6. $\frac{36}{5}x^5-12x^3+9x+C$
7. $\frac{1}{3}\ln|3x^2-5|+C$
8. $-\frac{1}{18}(3x^2-5)^{-6}+C$
9. $\frac{1}{6}(x^2+5)^6-\frac{1}{4}(x^2+5)^4+\frac{1}{2}(x^2+5)^2+C$
10. $\frac{1}{4}\ln|4t^2-3|+C$
11. $2e^{v^2-8}+C$
12. $\sec(p+1)+C$
13. $\frac{1}{3}\ln|3y-1|+C$
14. $-\frac{1}{2}\cos(h^4-2h)+C$
15. $e^{\tan r}+C$
16. $\ln|x^2-4|+\frac{3}{4}x^4-2x+C$

1.3.7 Homework: u-Substitution

Problem 1.25 Find $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$.

Problem 1.26 Find $\int \sec^2(6x-5)\,dx$.

Problem 1.27 Find $\int \frac{\sin(\ln x)}{x}\,dx$.

Problem 1.28 Find $\int \sin(\ln x)\,dx$.

Problem 1.29 Find $\int \left(\frac{1}{3x-4}+3x^5-2x+5\right)\,dx$.

Problem 1.30 Find $\int (3x^2-1)^2\,dx$.

Problem 1.31 Find $\int (6x+9)\sin(x^2+3x)\,dx$.

Problem 1.32 Find $\int (3x^2+5)^2\cdot 6x\,dx$.

Problem 1.33 Find $\int (3x^2+5)^2\,dx$.

Problem 1.34 Find $\int 3\cos(3x-1)e^{4\sin(3x-1)}\,dx$.

Problem 1.35 Find $\int 2xe^{x^2-10}\,dx$.

Problem 1.36 Find $\int (4x-5)(2x^2-5x)^{10}\,dx$.

Problem 1.37 Find $\int e^x(e^x-5)^{-3}\,dx$.

Problem 1.38 There are two solutions for the following. Obtain them both: $\int \cos x\sin x\,dx$.

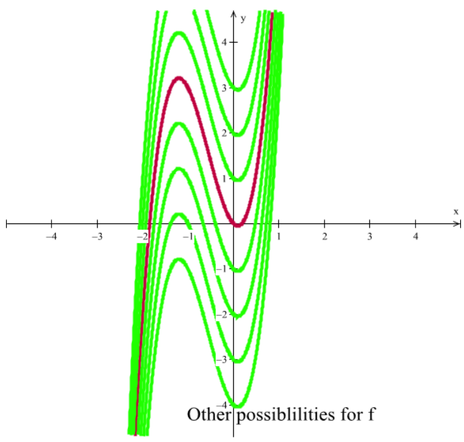
1.4 Slope Fields

Objectives — Slope Fields.

- Correlate slope fields to integrals and the constant of integration.
- Plot slope fields given a differential equation.

1.4.1 What is a Slope Field?

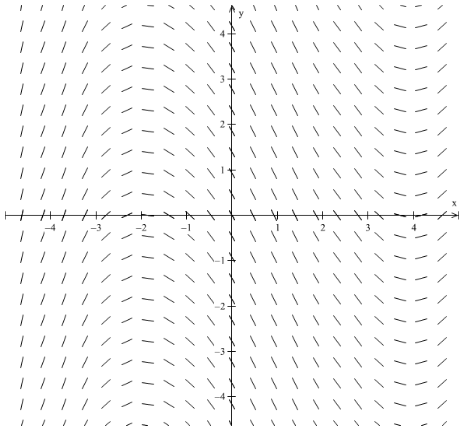
We can begin by thinking of a slope field is a graphical representation of the possible antiderivatives of a function. Essentially, consider a function $f(x)$ such that $F(x) = \int f(x)\,dx$. If we graphed the $F(x)$, we would have a slope field. It is called a slope field because recall that when we find antiderivatives, we're not sure what the constant of integration would be and so we don't know exactly where the graph will be. When we graph all of the possible answers, we end up with a "field" of graphs.



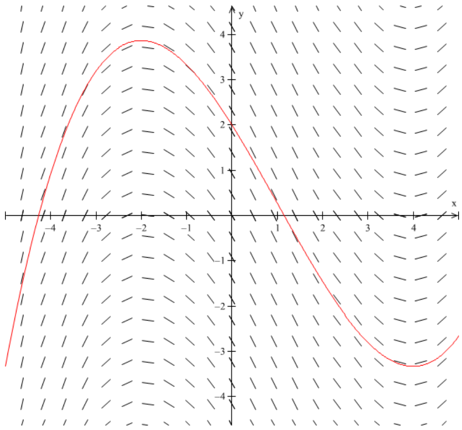
Other possiblilities for f

But, with a slope field, there's even a little more uncertainty. When we graph a slope field, not only are we unsure about where the graph is placed, we're not as certain about the graph itself. Rather than try to find the antiderivative of these equations, we will simply settle for knowing the slopes at various points for these equations. And instead of plotting a field of graphs, we will plot a field of slopes. Thus, slope field.

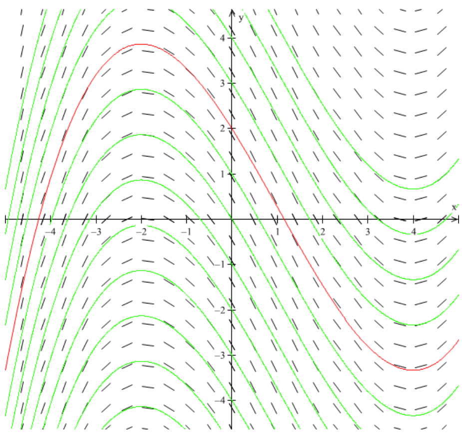
Vocabulary — Slope Field.
A slope field is simply a field of slopes that reveal the solution of a differential equation. They generally look like this:



If we were to choose a C and follow along the slopes, we might end up with a function like this:



Considering all of the possible options like we have in the past, we would have something like:



Vocabulary — Differential Equation.
We will discuss differential equations in more detail in Section 1.6, but for now, we can establish that a differential equation is an equation with differentials: $\mathrm{d}y$ or $\mathrm{d}x$ etc. which is of the form: $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y)$. For example, $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x+2y$ is a differential equation.

Recall that $\frac{\mathrm{d}y}{\mathrm{d}x}$ could also be written as y' and so, $\frac{\mathrm{d}y}{\mathrm{d}x}$ represents slope just as y' does.

1.4.2 Basic Differentials

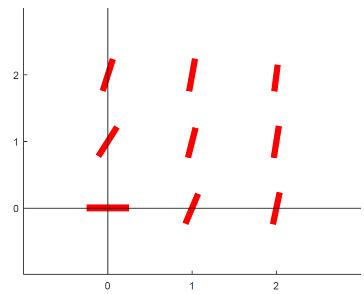
Let's begin by seeing some slope fields in action.

Example 1.35 Sketch the slope field for the differential equation $\frac{dy}{dx} = 3x + 2y$. ■

Our job will be to determine the slope at various points and then graph the slopes. Many people find it easiest to make a table. In future problems, you will be told which points to use, however, for this example, we will simply pick arbitrary points.

Point	Scratch Work	Slope
$(0,0)$	$\frac{dy}{dx} = 3(0)+2(0)$	0
$(0,1)$	$\frac{dy}{dx} = 3(0)+2(1)$	2
$(1,0)$	$\frac{dy}{dx} = 3(1)+2(0)$	3
$(1,1)$	$\frac{dy}{dx} = 3(1)+2(1)$	5
$(2,0)$	$\frac{dy}{dx} = 3(2)+2(0)$	6
$(2,1)$	$\frac{dy}{dx} = 3(2)+2(1)$	8
$(1,2)$	$\frac{dy}{dx} = 3(1)+2(2)$	7
$(0,2)$	$\frac{dy}{dx} = 3(0)+2(2)$	4
$(2,2)$	$\frac{dy}{dx} = 3(2)+2(2)$	10

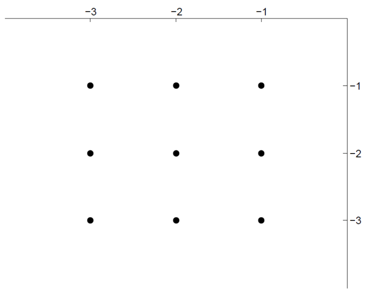
At $(0,0)$, we found out the slope should be 0, so we would go to the origin on a graph and make a short horizontal dash with that slope (flat). At $(2,2)$, we have a slope of 10, so we would go to the point (2.2) and draw a very steep line there. At $(1,1)$ where the slope is only 5, we would draw a slope about half as steep as the one we draw at $(2,2)$, and so on. We will go through all the points we chose in this example and draw small dashes with the right amount of steepness. When we finish, we get this which is our answer.



Notice that the three parallel hash marks we mentioned earlier are lined up down the diagonal of the graph. They actually provide some level of symmetry to the figure. These sorts of properties are important to notice as we complete the task.

AP Many students worry about getting the right slope drawn for each dash and even take extreme steps to ensure the dashes all have precisely the right amount of slope. However, on the AP test, they will keep it simple and are only looking to see that you understand the concept. A slope of 2 should be steeper than a slope of 1. Negative slopes should point down, positive slopes should point up. And horizontal and vertical slopes should be clear. Don't over-complicate it. Get the slopes close enough and you should be fine.

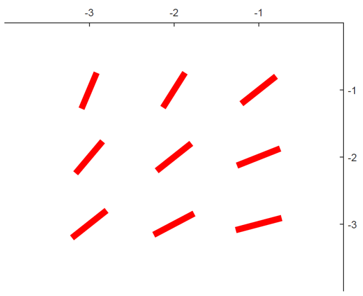
Example 1.36 Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{x}{y}$ on the graph given below. ■



In this case, we are given the points we should use. Notice that there are nine of them. We will set up a table with these nine points. The order we write them is not important.

Point	Scratch Work	Slope
$(-3,-1)$	$\frac{dy}{dx} = \frac{-3}{-1}$	3
$(-3,-2)$	$\frac{dy}{dx} = \frac{-3}{-2}$	1.5
$(-3,-3)$	$\frac{dy}{dx} = \frac{-3}{-3}$	1
$(-2,-1)$	$\frac{dy}{dx} = \frac{-2}{-1}$	2
$(-2,-2)$	$\frac{dy}{dx} = \frac{-2}{-2}$	1
$(-2,-3)$	$\frac{dy}{dx} = \frac{-2}{-3}$	0.67
$(-1,-1)$	$\frac{dy}{dx} = \frac{-1}{-1}$	1
$(-1,-2)$	$\frac{dy}{dx} = \frac{-1}{-2}$	0.5
$(-1,-3)$	$\frac{dy}{dx} = \frac{-1}{-3}$	0.33

Notice that we have several points here with a slope of 1. When we sketch this slope field, those should all be parallel. The hash mark at $(-3,-2)$ should be the steepest, the hash mark at $(-1,-3)$ should be the least steep. By sketching in each of these individual lines, we obtain:

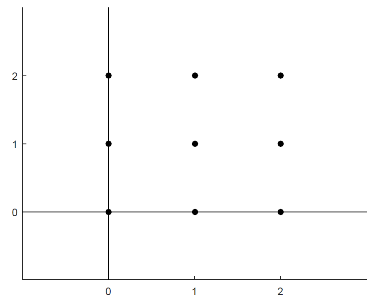


AP A very common mistake students make on the AP test is to not graph at each of the points. In most cases, you will be given a graph, as shown in the previous example, telling you where to plot each point. Before you begin anything, count the number of points showing. When you finish, count the number of hash marks you made and ensure you made the right amount.

1.4.3 Solving for $\frac{dy}{dx}$

In order to graph these plots, we must be able to calculated the slope at any point. In order to do this, we must always have $\frac{dy}{dx}$ isolated. If the problem does not begin that way, we must use Algebra to ensure it happens.

Example 1.37 Sketch the slope field for the differential equation $\frac{dy}{2dx} + \sqrt{x} = y$ on the graph given below. ■



This problem is different because $\frac{dy}{dx}$ is not by itself. That's easily addressed, though, we will simply use Algebra to solve for it.

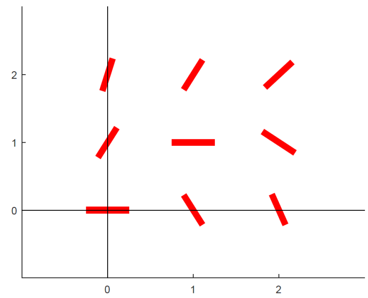
$$\frac{dy}{2dx} + \sqrt{x} = y$$
$$\frac{dy}{2dx} + \sqrt{x} - \sqrt{x} = y - \sqrt{x}$$
$$\frac{dy}{2dx} = y - \sqrt{x}$$

$$2 \cdot \frac{dy}{2dx} = 2 \cdot (y - \sqrt{x})$$
$$\frac{dy}{dx} = 2(y - \sqrt{x})$$

We can now easily solve for $\frac{dy}{dx}$ and plot is as per usual. We must be careful, as the y is on the left and x is on the right in the equation. This can be easily confuddled.

Point	Scratch Work	Slope
$(0,0)$	$2\left((0) - \sqrt{(0)}\right)$	0
$(1,0)$	$2\left((0) - \sqrt{(1)}\right)$	-2
$(2,0)$	$2\left((0) - \sqrt{(2)}\right)$	-2.83
$(0,1)$	$2\left((1) - \sqrt{(0)}\right)$	2
$(1,1)$	$2\left((1) - \sqrt{(1)}\right)$	0
$(2,1)$	$2\left((1) - \sqrt{(2)}\right)$	-0.83
$(0,2)$	$2\left((2) - \sqrt{(0)}\right)$	4
$(1,2)$	$2\left((2) - \sqrt{(1)}\right)$	2
$(2,2)$	$2\left((2) - \sqrt{(2)}\right)$	1.17

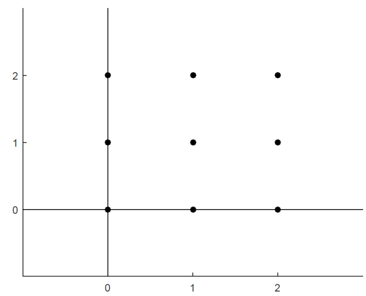
Sketching these as before, we have our answer:



1.4.4 Undefined Slopes

Thus far, we have only seen slopes that have been real numbers. How would we draw the slope field if we find a slope where we are dividing by zero? We simply sketch a vertical line at that point.

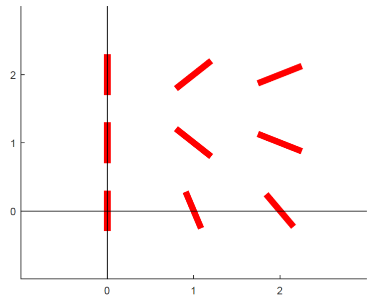
Example 1.38 Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{2y-3}{x}$ on the graph given below. ■



We begin as we usually do, by setting up a table with the points we need.

Point	Scratch Work	Slope
$(0,0)$	$\frac{2(0)-3}{(0)}$	undefined
$(1,0)$	$\frac{2(0)-3}{(1)}$	-3
$(2,0)$	$\frac{2(0)-3}{(2)}$	-1.5
$(0,1)$	$\frac{2(1)-3}{(0)}$	undefined
$(1,1)$	$\frac{2(1)-3}{(1)}$	-1
$(2,1)$	$\frac{2(1)-3}{(2)}$	-0.5
$(0,2)$	$\frac{2(2)-3}{(0)}$	undefined
$(1,2)$	$\frac{2(2)-3}{(1)}$	1
$(2,2)$	$\frac{2(2)-3}{(2)}$	0.5

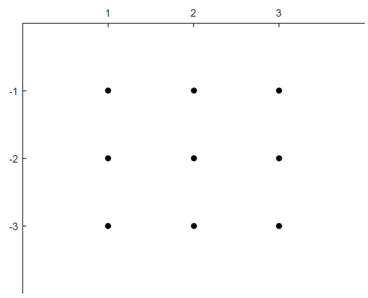
We see something a little strange here, we have three points that are undefined. Remember that a vertical line has an undefined slope. So at those points, we will make the hash marks vertical.



1.4.5 Differential Equations with One Variable

Occasionally, we will encounter a differential equation that has only x or y, but not both. Problems like these tend to confuse students because it seems as though information is missing. In fact, nothing is missing and this problem will actually be computationally easier than the other problems we have completed so far.

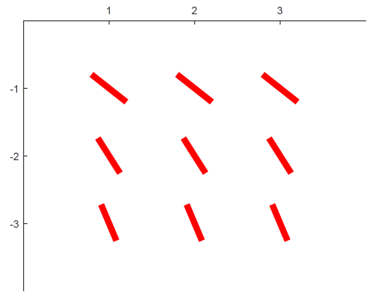
Example 1.39 Sketch the slope field for the differential equation $\frac{dy}{dx} = y$ on the graph given below. ■



Our table will be computationally easier because we only have one variable to concern us.

Point	Scratch Work	Slope
$(1,-1)$	(-1)	-1
$(1,-2)$	(-2)	-2
$(1,-3)$	(-3)	-3
$(2,-1)$	(-1)	-1
$(2,-2)$	(-2)	-2
$(2,-3)$	(-3)	-3
$(3,-1)$	(-1)	-1
$(3,-2)$	(-2)	-2
$(3,-3)$	(-3)	-3

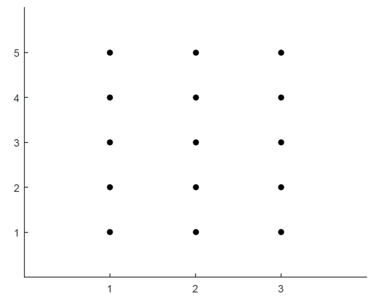
As we graph it, we actually see that the three columns will be identical. This is because the slopes do not depend on what the $x-$value is.



1.4.6 Finding Solutions

This is the last skill we will master in this section. We will not only sketch the slope field, but we will decide which of all of the graphs is the correct solution and sketch that as well.

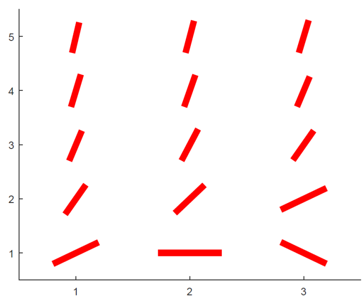
Example 1.40 Sketch the slope field for the differential equation $\frac{dy}{dx} = 2y - x$ for the graph given below, then sketch the solution that passes through the point $(2,2)$. ■



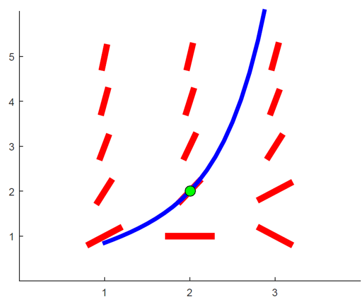
The beginning of this example will be the same as the ones that preceded it. However, we are being asked to do one extra step. We are being asked to sketch a specific solution at $(2,2)$. We'll begin with our table and plotting the slope field.

Point	Scratch Work	Slope
$(1,1)$	$2(1)-(1)$	1
$(1,2)$	$2(2)-(1)$	3
$(1,3)$	$2(3)-(1)$	5
$(1,4)$	$2(4)-(1)$	7
$(1,5)$	$2(5)-(1)$	9
$(2,1)$	$2(1)-(2)$	0
$(2,2)$	$2(2)-(2)$	2
$(2,3)$	$2(3)-(2)$	4
$(2,4)$	$2(4)-(2)$	6
$(2,5)$	$2(5)-(2)$	8
$(3,1)$	$2(1)-(3)$	-1
$(3,2)$	$2(2)-(3)$	1
$(3,3)$	$2(3)-(3)$	3
$(3,4)$	$2(4)-(3)$	5
$(3,5)$	$2(5)-(3)$	7

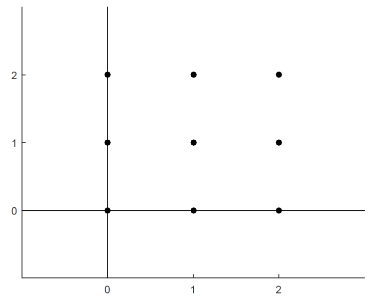
Graphing it, we see:



Now, to finish this problem, we need to sketch a solution through the point at $(2,2)$. We will plot that point on the graph and then sketch a curve that goes through the points as well as possible, as if we were playing "Connect the Dots." This will not be an exact science because we are hand-drawing it as opposed to performing it on a computer. However, our attempt to follow the slopes should be apparent.



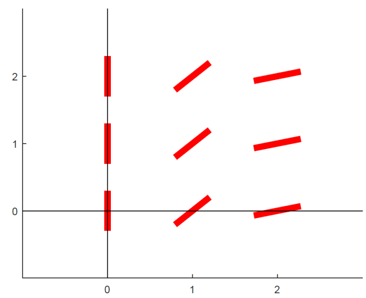
Example 1.41 Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{1}{x^2}$ for the graph given below, then sketch the solution that passes through the point $(1,2)$. ■



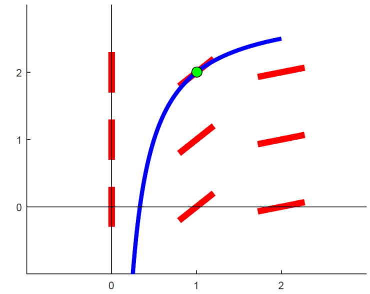
First, our table:

Point	Scratch Work	Slope
$(0,0)$	$\frac{1}{(0)^2}$	undefined
$(0,1)$	$\frac{1}{(0)^2}$	undefined
$(0,2)$	$\frac{1}{(0)^2}$	undefined
$(1,0)$	$\frac{1}{(1)^2}$	1
$(1,1)$	$\frac{1}{(1)^2}$	1
$(1,2)$	$\frac{1}{(1)^2}$	1
$(2,0)$	$\frac{1}{(2)^2}$	0.25
$(2,1)$	$\frac{1}{(2)^2}$	0.25
$(2,2)$	$\frac{1}{(2)^2}$	0.25

As we graph it, we notice that the rows are each identical to each other. This make sense, since the original differential equation, $\frac{dy}{dx} = \frac{1}{x^2}$ did not depend on the $y-$value.

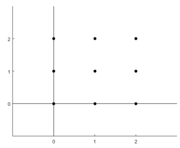


The only thing remaining is to sketch the solution that runs through the point $(1,2)$.

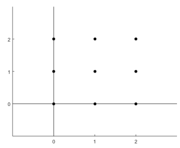


1.4.7 Practice Problems

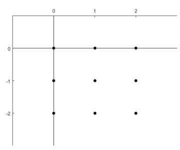
1. Sketch: $\frac{dy}{dx} = x - y$ on the grid below.



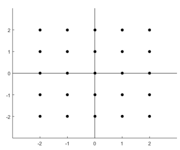
2. Sketch: $\frac{dy}{dx} = \frac{y+1}{x-2}$ on the grid below.



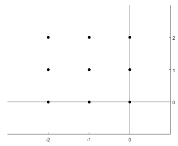
3. Sketch: $\frac{dy}{dx} = \sqrt{2x} - x$ on the grid below.



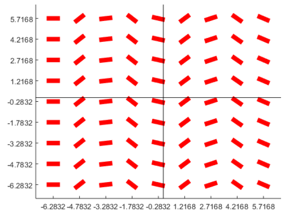
4. Sketch: $\frac{dy}{dx} = xy$ at $(1,-1)$.



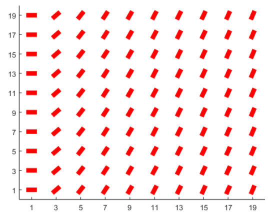
5. Sketch: $\frac{dy}{dx} = x^2 - y$ at $(-2,2)$.



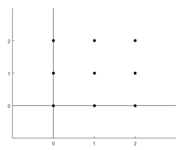
6. Sketch the solution at $(-2,2)$.



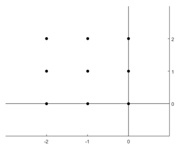
7. Sketch the solution at $(7,11)$.

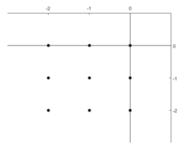


8. Sketch $\frac{dy}{dx}$ if $\left(\frac{dy}{dx}\right)^3 = x + 2y$.



9. Sketch $\frac{dy}{dx}$ if $\frac{dy}{dx} + 2 = x^2$.

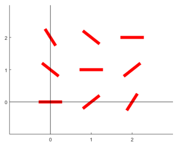




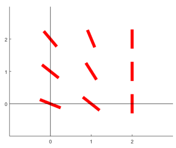
10. Sketch $\frac{dy}{dx}$ if $\frac{dy}{dx}+x=0$

1.4.8 Practice Problems Solutions

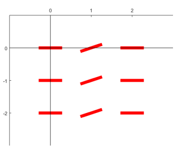
1. $\frac{dy}{dx}=x-y$



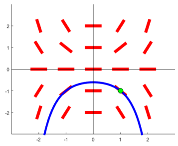
2. $\frac{dy}{dx}=\frac{y+1}{x-2}$



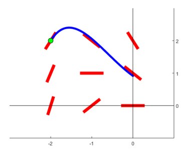
3. $\frac{dy}{dx}=\sqrt{2x}-x$



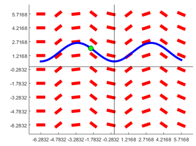
4. $\frac{dy}{dx}=xy$ at $(1,-1)$



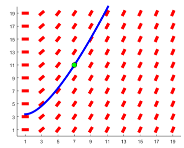
5. $\frac{dy}{dx}=x^2-y$ at $(-2,2)$



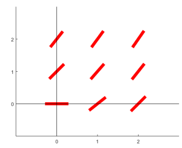
6. At $(-2,2)$



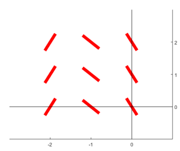
7. At $(7,11)$



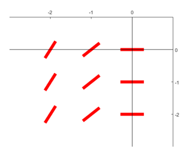
8. $\frac{dy}{dx}=\sqrt[3]{x+2y}$



9. $\frac{dy}{dx}=x^2-2$

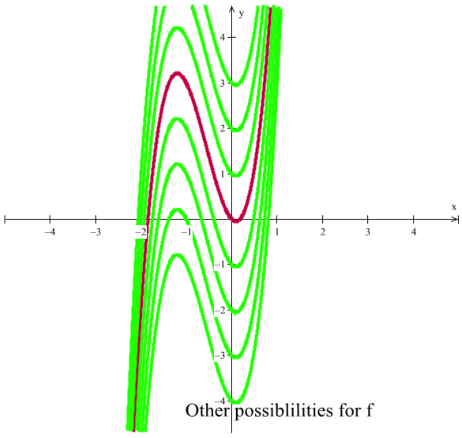


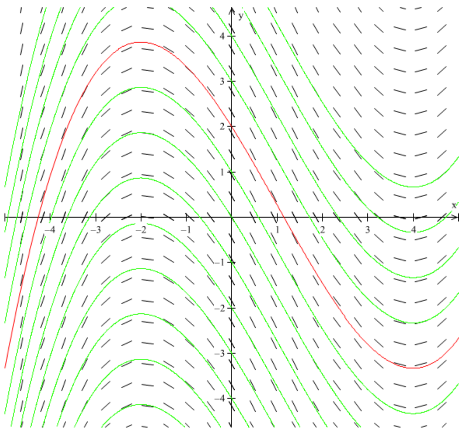
10. $\frac{dy}{dx}=-x$



1.4.9 A Word About the Constant of Integration

We have discussed this concept a few times now, that when we integrate or solve a differential equation, we have more than one possible solution because we don't yet know what that constant of integration could be. However, it is possible to choose one of the infinite answers as illustrated in the red curve in both of the figures below.





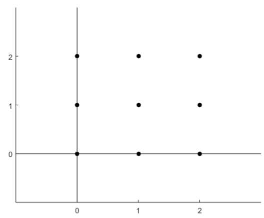
When we have a slope field and we are given a point to sketch a solution through, we are doing just that. We are choosing one of the infinite solutions that matches a particular scenario.

Remember, though, that each of the different solutions (those in green in the images above) differ only by the constant of integration. When C is larger, the solution may move higher up, etc. This means, then, that when we are given a point to investigate, we are in fact choosing a specific C.

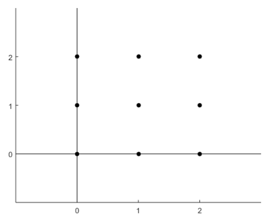
We will see in Section 1.5 how to solve a basic equation for the C. In Section 1.6, we will also be solving for C, however we will solve much more complicated differential equations.

1.4.10 Homework: Slope Fields

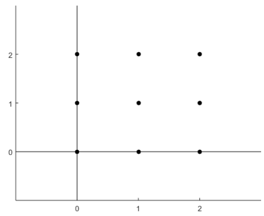
Problem 1.39 Sketch $\frac{dy}{dx}$ if $\frac{dy}{dx} - x + y^2 = 0$ on the grid below.



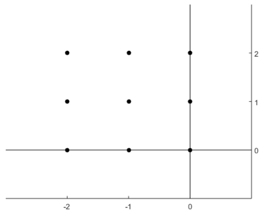
Problem 1.40 Sketch $\frac{dy}{dx} = 3$ on the grid below.



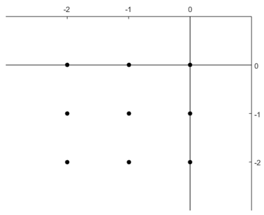
Problem 1.41 Sketch $\frac{dy}{dx} = x^2 + y$ and the solution that passes through $(0,0)$ on the grid below.



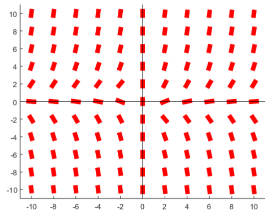
Problem 1.42 Sketch $\frac{dy}{dx}$ if $\frac{dy}{dx}+2x=0$ on the grid below. Then, sketch the solution at $(-1,2)$.



Problem 1.43 Sketch $\frac{dy}{dx}=\frac{3}{y^2}$ on the grid below.



Problem 1.44 On the graph shown below, sketch the solution at $(2,1)$.



1.5 Solving for C

Objectives — Solving for C.

- Use initial conditions and boundary conditions to solve for C, the constant of integration.

1.5.1 Boundary and Initial Conditions

In Section 1.4 we learned how to "solve" a differential equation graphically. We will now learn how to solve it analytically. We will be given specific points just as we did before, but now we will actually use them to solve for C as opposed to simply sketching the graph. This will be an important skill for later when we see word problems.

Vocabulary — Initial Conditions.

Initial conditions describe the state of a system initially. For example, if the height of a particle is being studied with respect to **time**, and the study of this particle begins when the particle is at a height of 100, then the initial condition would be that at $t = 0$, $y = 100$ where t represents time and y represents the height of the particle. This is frequently written as $y(0) = 100$ or $(0, 100)$. Initial conditions are generally dependent on time.

Vocabulary — Boundary Conditions.
Boundary Conditions are very similar to initial conditions, but refer to the state of a system at a certain point. For example, if the height of a particle is being studied with respect to where it is located along the $x-$axis (the **position**), and at the beginning of its journey, it already has a height of -2, then the boundary condition would be at $x = 0$, $y = -2$ where t represents time and x represents the position of the particle along the $x-$axis. This is frequently written as $y(0) = -2$ or $(0, -2$. Boundary conditions are generally dependent on position.

Boundary and **initial** conditions are important because they help us to solve for the constant of integration, C.

There is an important lesson to be learned here: a particle's height can be dependent on time or on position, depending on the circumstances and what is being studied. It is even possible for the particle's height to be dependent on both time and position simultaneously which might be addressed in more advanced Math classes.

It is also important to note that in general, the distinction between an initial condition and a boundary condition is simply based on the distinction between time or position. In this section, both will mathematically be handled identically.

The last important note to consider is that the initial conditions and boundary conditions do not always have to be given with 0, however, they usually are.

Example 1.42 Find a solution for y when $y = \int (3x+2)\,\mathrm{d}x$ that passes through $(0,2)$. ■

The first part of this will be integrating just as we did in the first three sections of this book.

$$\int (3x+2)\,\mathrm{d}x = \frac{3}{2}x^2 + 2x + C$$

Ordinarily, we would leave the C alone and call $y = \frac{3}{2}x^2 + 2x + C$ our answer. However, at this point, we have been given a boundary condition we can work with, $(0,2)$. We will plug this in to solve for C. note that this point tells us that when $x = 0$ then $y = 2$.

$$y = \frac{3}{2}x^2 + 2x + C$$
$$(2) = \frac{3}{2}(0)^2 + 2(0) + C \; 2 = C$$

By using this known point, we have solved for C and narrowed down all of the possible solutions to one single answer which is: $y = \frac{3}{2}x^2 + 2x + 2$.

1.5.2 Working with Differentials

Example 1.43 Find a solution for y when $2\cos x\,\mathrm{d}x = \mathrm{d}y$ if $y\left(\frac{\pi}{2}\right) = 4$. ■

This question is phrased a little differently that before. Specifically, we see that there is no integral, and there is also no y. This is no problem, though, we can just integrate both sides and then be back into familiar territory.

$$2\cos x\,dx = dy$$
$$\int 2\cos x\,dx = \int dy$$
$$2\sin x + C_1 = y + C_2$$

Because we are doing two separate integrals, we need to have a constant of integration on each side. Since these could be totally different numbers, we will use C_1 and C_2 to keep them distinct. Let's manipulate a little bit:

$$2\sin x + C_1 - C_2 = y$$

We now have an expression for y, all that remains is solving for the constants. For the moment, we'll ignore that there are two of them, and simply follow the same process we did in the last example. Reading about the boundary condition, $y\left(\frac{\pi}{2}\right) = 4$, we conclude that when $x = \frac{\pi}{2}$, then $y = 4$.

$$2\sin\left(\frac{\pi}{2}\right) + C_1 - C_2 = (4)$$
$$2\cdot(1) + C_1 - C_2 = 4$$
$$2 + C_1 - C_2 - 2 = 4 - 2$$
$$C_1 - C_2 = 2$$

Notice that we didn't actually solve for C_1 and C_2 separately. It doesn't really matter what they each are. All we're concerned with is the ability to write an expression for y without any constants and we have accomplished this: $y = 2\sin x + 2$

Example 1.44 Find a solution for y by solving the differential equation $(2x-5)^2\frac{dx}{dy} = 1$ when $y(2) = 3$ ■

Again, this looks a little different because there's no y by itself. No problem, we can rearrange. Treat the $\frac{dx}{dy}$ just like any fraction.

$$(2x-5)^2\frac{dx}{dy} = 1$$
$$(2x-5)^2\frac{dx}{dy}\cdot dy = 1\cdot dy$$
$$(2x-5)^2\,dx = dy$$

Now, we're ready to integrate. We'll have to use $u-$substitution on the left which is left to the reader to check.

$$\int (2x-5)^2\,dx = \int dy$$
$$\frac{1}{2}\cdot\frac{1}{3}(2x-5)^3 + C_1 = y + C_2$$
$$\frac{1}{2}\cdot\frac{1}{3}(2x-5)^3 + C_1 - C_2 = y$$
$$\frac{1}{6}(2x-5)^3 + C_1 - C_2 = y$$

Using our boundary condition:

$$\frac{1}{6}(2(2)-5)^3 + C_1 - C_2 = 3$$
$$\frac{1}{6}(-1)^3 + C_1 - C_2 = 3$$
$$-\frac{1}{6} + C_1 - C_2 = 3$$
$$C_1 - C_2 = 3 + \frac{1}{6}$$
$$C_1 - C_2 = \frac{19}{6}$$

We are ready to write our answer: $y = \frac{1}{6}(2x-5)^3 + \frac{19}{6}$.

Example 1.45 Find a solution for y by solving the differential equation $\sin(2x) = \frac{dy}{dx}$ with the boundary condition $(-\pi, 0)$. ■

As with all of the other problems, we'd like to have dy be by itself before we begin to integrate. We will need to use $u-$substitution for the actual integration.

$$\begin{aligned} \sin 2x &= \frac{dy}{dx} \\ \sin 2x \cdot dx &= \frac{dy}{dx} \cdot dx \\ \sin 2x\, dx &= dy \end{aligned}$$

$$\begin{aligned} \int \sin(2x)\, dx &= \int dy \\ -\tfrac{1}{2}\cos(2x) + C_1 &= y + C_2 \\ -\tfrac{1}{2}\cos(2x) + C_1 - C_2 &= y \end{aligned}$$

$$\begin{aligned} -\tfrac{1}{2}\cos(2(-\pi)) + C_1 - C_2 &= (0) \\ -\tfrac{1}{2}\cos(-2\pi) + C_1 - C_2 &= y \\ -\tfrac{1}{2}(1) + C_1 - C_2 &= 0 \\ C_1 - C_2 &= \tfrac{1}{2} \end{aligned}$$

We have arrived at our solution, $y = -\frac{1}{2}\cos(2x) + \frac{1}{2}$.

1.5.3 Working with the Second Derivative

Example 1.46 Find an expression for y if $\frac{d^2y}{dx^2} = 2x - 1$, $\frac{dy(0)}{dx} = 4$ and $y(0) = 2$. ■

Recall that $\frac{d^2y}{dx^2}$ means the second derivative of y. This means that in order to get to y, we'll need to integrate twice. This is why we have two boundary conditions, one for each integral step. We note that we can separate $\frac{d^2y}{dx^2}$ to $\frac{d}{dx} \cdot \frac{dy}{dx}$. (Consider that in the numerator, d^2y, we can think of only the d being squared. However, in the denominator, dx^2, it is actually the entire dx that is begin squared).

$$\begin{aligned} \frac{d^2y}{dx^2} &= 2x - 1 \\ \frac{d}{dx} \cdot \frac{dy}{dx} &= 2x - 1 \\ \frac{d}{dx} \cdot \frac{dy}{dx} \cdot dx &= (2x-1) \cdot dx \\ \frac{d}{dx} \cdot dy &= (2x-1)\, dx \end{aligned}$$

$$\begin{aligned} \int \frac{d}{dx} \cdot dy &= \int (2x-1)\, dx \\ \frac{d}{dx} \int dy &= \int (2x-1)\, dx \\ \frac{d}{dx} y + C_1 &= x^2 - x + C_2 \\ \frac{d}{dx} y &= x^2 - x + C_2 - C_1 \\ \frac{dy}{dx} &= x^2 - x + C_2 - C_1 \end{aligned}$$

$$\begin{aligned} 4 &= 0^2 - 0 + C_2 - C_1 \\ C_2 - C_1 &= 4 \\ \frac{dy}{dx} &= x^2 - x + 4 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} \cdot dx &= (x^2 - x + 4) \cdot dx \\ dy &= (x^2 - x + 4)\, dx \end{aligned}$$

$$\int dy = \int (x^2 - x + 4)\, dx$$

$$y + C_1 = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 4x + C_2$$
$$y = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 4x + C_2 - C_1$$

$$2 = \frac{1}{3}(0)^3 - \frac{1}{2}(0)^2 + 4(0) + C_2 - C_1$$
$$C_2 - C_1 = 2$$

We have finally arrived at our answer: $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 4x + 2$.

While correct, the notation for this example was tedious, even in the author's opinion. In the future, we'll not worry so much about using perfect notation and focus instead on getting the correct answer.

Example 1.47 Find an expression for y given that $\frac{d^2y}{dx^2} = \cos x$, $\frac{dy(0)}{dx} = 5$, and $y(0) = 3$. ■

As promised, we will focus more on getting the answer and leave some of the notation out. Note first that we can rewrite $\frac{d^2y}{dx^2}$ as simply y'' and $\frac{dy(0)}{dx} = 1$ as $y'(0) = 5$. This has the potential to simplify our scratch work considerably.

$$\frac{d^2y}{dx^2} = \cos x$$
$$y'' = \cos x$$

We will integrate both sides without necessarily writing out all of the integral symbols and differentials.

$$y'' = \cos x$$
$$y' + C_1 = \sin x + C_2$$
$$y' = \sin x + C_2 - C_1$$

$$(5) = \sin(\pi) + C_2 - C_1$$
$$C_2 - C_1 = 5$$
$$y' = \sin x + 5$$

$$y = -\cos x + 5x + C_2 - C_1$$
$$(3) = -\cos(\pi) + 5(\pi) + C_2 - C_1$$
$$3 = -(-1) + 5\pi + C_2 - C_1$$
$$2 - 5\pi = C_2 - C_1$$

We have arrived at our answer: $y = -\cos x + 5x + 2 - 5\pi$. We can see that using y' instead of the notation with differentials made this problem slightly simpler.

1.5.4 Using an Initial Condition

Example 1.48 Let $y'' = e^t + 1$ and use the initial conditions $y'(0) = -2$ and $y(2) = 4$. Find a solution for $y(t)$. ■

The only substantial difference in this problem is that we are now using t instead of x. In this case, the conditions might more accurately be referred to as initial conditions instead of boundary conditions. We will proceed exactly as we did in the last example.

$$y'' = e^t + 1$$
$$y' = e^t + t + C_2 - C_1$$
$$(-2) = e^{(0)} + (0) + C_2 - C_1$$
$$C_2 - C_1 = -2$$

$$y' = e^t + t - 2$$

$$\begin{aligned} y &= e^t + \tfrac{1}{2}t^2 - 2t + C_2 - C_1 \\ (4) &= e^(2) + \tfrac{1}{2}(2)^2 - 2(2) + C_2 - C_1 \\ 6 - e^2 &= C_2 - C_1 \end{aligned}$$

So, our solution is $y = e^t + \frac{1}{2}t^2 - 2t + 6 - e^2$

1.5.5 Practice Problems

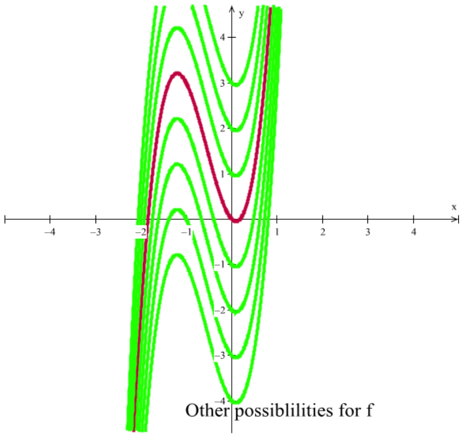
1. Solve for y given the initial condition $(1,4)$ if $y = \int \frac{1}{t}\,\mathrm{d}t$.
2. Find a solution for y if $\mathrm{d}y = \frac{1}{x^2}\,\mathrm{d}x$ that passes through the point $\left(4, \frac{1}{4}\right)$.
3. Solve the differential equation $\sec^2 x \frac{\mathrm{d}x}{\mathrm{d}y} = 3$ to find a solution for y with the boundary condition $y\left(\frac{\pi}{4}\right) = 5$.
4. Let $2x\sin(x^2)\,\mathrm{d}x = \mathrm{d}y$. If $y(0) = -3$, solve for y.
5. For the differential equation $\sqrt{x} = \frac{\mathrm{d}y}{\mathrm{d}x}$, solve for the solution in which $y(0) = 9$.
6. If $\frac{1}{x^2} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, $\frac{\mathrm{d}y(1)}{\mathrm{d}x} = 4$, and $y(1) = 2$, find the solution for y.
7. Find the solution for y with initial condition $y(5) = -6$ if $y = \int e^t\,\mathrm{d}t$.
8. Solve the differential equation $y'' = \sin x$ with boundary conditions $y'(0) = 4$ and $y(0) = 3$.
9. Let $\sqrt{t+2} = \frac{\mathrm{d}y}{\mathrm{d}t}$ and let $y = 5$ when $t = -2$. Find the solution for y.
10. Solve the differential equation $x^3 \frac{\mathrm{d}x}{\mathrm{d}y} = 1$ at $y(2) = -1$.

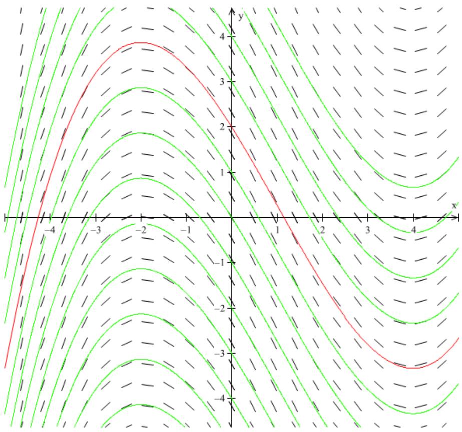
1.5.6 Practice Problems Solutions

1. $y = \ln|x| + 4$
2. $y = -x^{-1} + \frac{1}{2}$
3. $y = \frac{1}{3}(\tan x + 14)$
4. $y = -\cos(x^2) - 2$
5. $y = \frac{2}{3}x^{\frac{3}{2}} + 9$
6. $y = -\ln|x| + 4x - 2$
7. $y = e^t - 6 - e^5$
8. $y = -\sin x + 5x + 3$
9. $y = \frac{2}{3}(x+2)^{\frac{3}{2}} + 5$
10. $y = \frac{1}{4}x^4 - 5$

1.5.7 A Word About the Constant of Integration

In section 1.4, we discussed the images below and we noted that our approach was to find a given point on the graph and then sketch the curve that followed the slopes and passed through that point. Doing this narrows down the possible solutions from infinitely many to one single solution.





In this section, we accomplished the same mission, however we did it analytically as opposed to graphically. As soon as we choose a C, we have narrowed down the possible solutions from infinitely many to one.

1.5.8 Homework: Solving for c

Problem 1.45 Find a solution for y that passes through $(2,-1)$ by solving the differential equation $2xe^{x^2} = \frac{dy}{dx}$.

Problem 1.46 If $2e^x = \frac{d^2y}{dx^2}$, find y when $y'(0) = 4$ and $y(0) = 3$.

Problem 1.47 Let $1 = \csc^2 x \frac{dx}{dy}$ and $y\left(\frac{\pi}{4}\right) = 1$, what is the solution for y?

Problem 1.48 Let y be given by $y = \int (3x-4)\,dx$. Find the solution that passes through the point $(2,5)$

Problem 1.49 The second derivative of y is given by $y'' = 6t$ and the initial conditions are given as $y'(1) = 4$ and $y(1) = -2$. Find the solution for y.

Problem 1.50 The slope of a function is described as $y' = \frac{1}{t} + 2t$. What is the solution of this function with the initial condition $y(1) = 4$?

Problem 1.51 Solve the differential equation $\frac{dy}{dx} = 24(4x+2)^2$ with the boundary condition $y(0) = 12$.

Problem 1.52 A function y has a boundary condition $y\left(\frac{1}{4}\right) = -2$ and is described as $y = \int e^{4x}\,dx$. Solve for y.

1.6 Separation of Variables

When learning about slope fields in Section 1.4, we graphed some differential equations that would be very difficult – maybe even impossible – to solve analytically. In Section 1.5, we solved differential equations that were a little more straight-forward than the ones featured in Section 1.4 because they only ever had the independent variable (x) and the differentials $\left(\frac{dy}{dx}\right)$.

Now, we will begin to solve some more difficult problems where we will see $\frac{dy}{dx}$, x, and also y. These quickly become very difficult; we will only be focusing on one technique – separation of variables.

Objectives — Separation of Variables.

- Use the Separation of Variables technique to set differential equations up for solving.
- Solve differential equations given an initial condition.

Vocabulary — Separation of Variables.
Separation of Variables is one of the most powerful and used techniques in solving differential equations. The technique works with the assumption that a differential equation can be separated into individual factors for each of the variables.

Vocabulary — Differential Equation.
A differential equation is a relationship between a function y and its derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$. Considering the example of a car in motion, it is an equation that represents the car's position with its velocity that could look like: $\frac{\mathrm{d}y}{\mathrm{d}t} = ty$ where t represents the time, y represents the position, and $\frac{\mathrm{d}y}{\mathrm{d}t}$ represents the velocity.

Theorem 1.6.1 — Separation of Variables. The technique of Separation of variables requires rearranging the non-differential side of an equation into factors for each of the variables like this: $\frac{\mathrm{d}y}{\mathrm{d}t} = Y(y)T(t)$. This can then be separated as: $\frac{\mathrm{d}y}{Y(y)} = \mathrm{d}t \cdot T(t)$ which can usually be easily solved.

Example 1.49 Let f be a function $y = f(t)$ where $\frac{\mathrm{d}y}{\mathrm{d}t} = 3yt$ and $f(3) = 5$. Find $y = f(t)$ by solving the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t}$. ■

As the name suggests, we always want to begin by separating the variables in the way described in Theorem 1.6.1. To match up with what we saw above, $Y(y) = y$ and $T(t) = 3t$ in this case. We can think of this algebraically and can accomplish this by simply multiplying both sides by $\mathrm{d}t$ and dividing both sides by y.

$$\mathrm{d}t \cdot \frac{\mathrm{d}y}{\mathrm{d}t} = 3yt \cdot \mathrm{d}t$$
$$\mathrm{d}y = 3yt\,\mathrm{d}t$$

$$\frac{\mathrm{d}y}{y} = \frac{3yt\,\mathrm{d}t}{y}$$
$$\frac{1}{y}\,\mathrm{d}y = 3t\,\mathrm{d}t$$

Our end goal is to get rid of the differentials, $\mathrm{d}y$ and $\mathrm{d}t$. The best way to approach this is to integrate. Integration is just like anything else in Algebra. Whatever you do to one side, you must do to the other. Therefore, we'll integrate both sides. Remember as we do this, these are indefinite integrals, so we will need to use the constant of integration. Since we have two different sides, we'll have two different constants of integration (they could both be different numbers). So, we'll use C_1 and C_2 so we can keep them straight.

$$\int \frac{1}{y}\,\mathrm{d}y = \int 3t\,\mathrm{d}t$$
$$\ln|y| + C_1 = \frac{3}{2}t^2 + C_2$$

We want to solve for y, eventually, so let's begin by moving C_1 over to the other side:

$$\ln|y| = \frac{3}{2}t^2 + C_2 - C_1$$

Consider, though, that $C_2 - C_1$ will still just be some constant. There's really no reason to continue writing out both of them. We can instead just write:

$$\ln|y| = \frac{3}{2}t^2 + C$$

where C represents $C_2 - C_1$.

Most people actually skip straight to this step, as it's somewhat predictable. They simply write:

$$\int \frac{1}{y}\,\mathrm{d}y = \int 3t\,\mathrm{d}t$$
$$\ln|y| = \frac{3}{2}t^2 + C$$

and simply forget about the C_1 and C_2 business.

Because they gave us an initial condition, that means we need to solve for C. We'll use the same techniques used in Section 1.5. We are told that $f(3) = 5$, so plugging this in and solving for C:

$$\ln|5| = \frac{3}{2}3^2 + C$$
$$C = \frac{27}{2} - \ln|5|$$

Because this is quite long, I will simply state that $C = \frac{27}{2} - \ln|5|$ and leave it written as C.

Now, to solve for y, we must get rid of the natural log. The inverse of natural log is exponentiating both sides with Euler's number. Doing that we have:

$$e^{\ln|y|} = e^{\frac{3}{2}t^2 + C}$$
$$y = e^{\frac{3}{2}t^2 + C}$$

Notice that the absolute value symbols around y are no longer necessary. This is because it is not possible for $e^{\frac{3}{2}t^2+C}$ to be negative, so we don't have to worry about y being negative.

Remembering our Algebra, we know that $e^{a+b} = e^a e^b$ and we use this to simplify further:

$$y = e^{\frac{3}{2}t^2} e^C$$

Recall from section 1.5 that even though e^C is just going to be some other constant, we cannot rewrite this as:

$$y = Ce^{\frac{3}{2}t^2}$$

because we have already solved for C.

Believe it or not, we have completed this problem, We can say that $y = e^C e^{\frac{3}{2}t^2}$ where $C = \frac{27}{2} - \ln(5)$.

This problem certainly had a lot of steps and perhaps they were not all obvious. Fortunately, there are a set of steps we can simply stick to when solving these types of problems, there is really very little imagination involved here. With a little practice, these problems will eventually become very quick.

AP You will most likely find this question on the free response section of the test. There are generally five steps you can follow:

1. Separate the variables
2. Integrate both sides
3. Add C to one side
4. Solve for C
5. Solve for y

1.6.1 Using the Five Steps

Example 1.50 Consider a function y where $\frac{dy}{dx} = \frac{3x}{y}$ and $y(0) = 2$. Solve the differential equation to obtain an equation for y. ■

We will simply follow the five steps we laid out in the AP Remark:

Step 1: Separate the variables

$$dx \cdot \frac{dy}{dx} = \frac{3x}{y} \cdot dx$$
$$dy = \frac{3x}{y}\,dx$$

$$dy \cdot y = \frac{3x}{y}\,dx \cdot y$$
$$y\,dy = 3x\,dx$$

Step 2: Integrate both sides

$$\int y\,dy = \int 3x\,dx$$
$$\frac{1}{2}y^2 = \frac{3}{2}x^2$$

Step 3: Add C to one side

$$\frac{1}{2}y^2 = \frac{3}{2}x^2 + C$$

Step 4: Solve for C

$$\frac{1}{2}(2)^2 = \frac{3}{2}(0)^2 + C$$
$$\frac{4}{2} = 0 + C$$
$$C = 2$$
$$\frac{1}{2}y^2 = \frac{3}{2}x^2 + 2$$

Step 5: Solve for y

$$\frac{1}{2}y^2 = \frac{3}{2}x^2 + 2$$
$$2 \cdot \frac{1}{2}y^2 = 2 \cdot \left(\frac{3}{2}x^2 + 2\right)$$
$$y^2 = 3x^2 + 4$$
$$y = \pm\sqrt{3x^2 + 4}$$

So we have our answer: $y = \pm\sqrt{3x^2 + 4}$

Example 1.51 If $\frac{dy}{dx} = y \cdot \sin x$ and $y(\pi) = 1$, obtain y by solving the differential equation. ■

Just as you will always do with these types of problems, you will follow the five steps:

Step 1: Separate the variables

$$\frac{dy}{dx} \cdot dx = y \cdot \sin x \cdot dx$$
$$dy = y \cdot \sin x\,dx$$

$$\frac{dy}{y} = \frac{y \cdot \sin x\,dx}{y}$$
$$\frac{dy}{y} = \sin x\,dx$$

Step 2: Integrate both sides

$$\int \frac{dy}{y} = \int \sin x \, dx$$
$$\ln|y| = -\cos x$$

Step 3: Add C to one side

$$\ln|y| = -\cos x + C$$

Step 4: Solve for C

$$\ln|1| = -\cos \pi + C$$
$$0 = -(-1) + C$$
$$C = -1$$
$$\ln|y| = -\cos x - 1$$

Step 5: Solve for y

$$\ln|y| = -\cos x - 1$$
$$e^{\ln|y|} = e^{-\cos x - 1}$$
$$y = e^{-1} e^{-\cos x}$$

So, our final answer is: $y = e^{-1} e^{-\cos x}$

Example 1.52 Solve the differential equation $\frac{dy}{dx} = y^2\sqrt{x}$ if $y(0) = 9$. ■

Always follow the five steps:

Step 1: Separate the variables

$$\frac{dy}{dx} = y^2\sqrt{x}$$
$$dx \cdot \frac{dy}{dx} = y^2\sqrt{x} \cdot dx$$
$$dy = y^2\sqrt{x}\, dx$$

$$\frac{dy}{y^2} = \frac{y^2\sqrt{x}\, dx}{y^2}$$
$$y^{-2}\, dy = x^{\frac{1}{2}}\, dx$$

Step 2: Integrate both sides

$$\int y^{-2}\, dy = \int x^{\frac{1}{2}}\, dx$$
$$-y^{-1} = \frac{2}{3}x^{\frac{3}{2}}$$

Step 3: Add C to one side

$$-y^{-1} = \frac{2}{3}x^{\frac{3}{2}} + C$$

Step 4: Solve for C

$$-(9)^{-1} = \frac{2}{3}(0)^{\frac{3}{2}} + C$$
$$C = -\frac{1}{9}$$
$$-y^{-1} = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{9}$$

Step 5: Solve for y

$$-y^{-1} = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{9}$$
$$y^{-1} = -\frac{2}{3}x^{\frac{3}{2}} + \frac{1}{9}$$
$$(y^{-1})^{-1} = \left(-\frac{2}{3}x^{\frac{3}{2}} + \frac{1}{9}\right)^{-1}$$
$$y = \frac{1}{-\frac{2}{3}x^{\frac{3}{2}} + \frac{1}{9}}$$

So, our final answer is $y = \frac{1}{-\frac{2}{3}x^{\frac{3}{2}} + \frac{1}{9}}$.

1.6.2 Determining when Separation of Variables Fails

Example 1.53 Solve the differential equation $\frac{dy}{dx} = x + y$ if $y(3) = 2$. ■

Let's follow our same steps:

Step 1: Separate the variables We seem to have a problem here. We are supposed to rewrite the right side as $X(x)Y(y)$, there is no way to rearrange the x and the y so that they are being multiplied. Because separation of variables is the only technique we have learned so far to solve differential equations, we do not have the mathematical tools to deal with this one. You will learn how to do this in future Math classes.

Example 1.54 Solve the differential equation $\frac{dx}{dt} = t + xt$ if $x(0) = -2$. ■

At first glance, this appears to have the same problem as Example 1.53. However, let's have a closer look. What if we factor out a t on the right-hand side. We would have:

$$\frac{dx}{dt} = t(1+x)$$

We actually now have what we needed. Thinking about $X(x)T(t)$, we can see that $T(t) = t$ and $X(x) = 1 + x$. We are ready to proceed with our steps.

Step 1: Separate the variables

$$\begin{aligned} \frac{dx}{dt} &= t(1+x) \\ dt \cdot \frac{dx}{dt} &= t(1+x) \cdot dt \\ dx &= t(1+x)\,dt \end{aligned}$$

$$\begin{aligned} \frac{dx}{1+x} &= \frac{t(1+x)\,dt}{1+x} \\ \frac{dx}{1+x} &= t\,dt \end{aligned}$$

Step 2: Integrate both sides (use $u-$substitution)

$$\begin{aligned} \int \frac{dx}{1+x} &= \int t\,dt \\ \ln|1+x| &= \frac{1}{2}t^2 \end{aligned}$$

Step 3: Add C to one side

$$\ln|1+x| = \frac{1}{2}t^2 + C$$

Step 4: Solve for C

$$\begin{aligned} \ln|1+(-2)| &= \frac{1}{2}(0)^2 + C \\ C = \ln|-1| &= \ln(1) = 0 \\ \ln|1+x| &= \frac{1}{2}t^2 + 0 \end{aligned}$$

Step 5: Solve for x

$$\begin{aligned} \ln|1+x| &= \frac{1}{2}t^2 + 0 \\ e^{\ln|1+x|} &= e^{\frac{1}{2}t^2} \\ 1+x &= e^{\frac{1}{2}t^2} \\ x &= e^{\frac{1}{2}t^2} - 1 \end{aligned}$$

We have finished: $x = e^{\frac{1}{2}t^2} - 1$.

AP How do you know which variable to solve for? Look at the differentials. If you have: $\frac{dJ}{dr}$, solve for J. If you have: $\frac{d(elephant)}{d(banana)}$, solve for elephant. In short, always solve for the variable on top. The variable on top is your dependent variable and the one on bottom is your independent variable.

1.6.3 Differentials with One Variable

Example 1.55 Solve the differential equation $\frac{dV}{dp} = k\sqrt{V}$ if $V(2) = 3$. ■

We observe something odd with this problem, the independent variable, p, is nowhere to be seen on the right-hand side. This is no typo. k is acting as a constant and there is no independent variable. That is not required. We will proceed as we always have.

Step 1: Separate the variables

$$\frac{dV}{dp} = k\sqrt{V}$$
$$dp \cdot \frac{dV}{dp} = k\sqrt{V} \cdot dp$$
$$dV = k\sqrt{V}\,dp$$

$$\frac{dV}{\sqrt{V}} = \frac{k\sqrt{V}\,dp}{\sqrt{V}}$$
$$V^{-\frac{1}{2}}\,dV = k\,dp$$

Step 2: Integrate both sides

$$\int V^{-\frac{1}{2}}\,dV = \int k\,dp$$
$$2V^{\frac{1}{2}} = kp$$

Step 3: Add C to one side

$$2V^{\frac{1}{2}} = kp + C$$

Step 4: Solve for C

$$2(3)^{\frac{1}{2}} = k(2) + C$$
$$C = \sqrt{3} - 2k$$
$$2V^{\frac{1}{2}} = kp + \sqrt{3} - 2k$$

Step 5: Solve for V

$$\frac{1}{2} \cdot 2V^{\frac{1}{2}} = \frac{1}{2} \cdot (kp + \sqrt{3} - 2k)$$
$$V^{\frac{1}{2}} = \frac{1}{2} \cdot (kp + \sqrt{3} - 2k)$$
$$\left(V^{\frac{1}{2}}\right)^2 = \left(\frac{1}{2} \cdot (kp + \sqrt{3} - 2k)\right)^2$$
$$V = \left(\frac{1}{2} \cdot (kp + \sqrt{3} - 2k)\right)^2$$

We finish with our answer $V = \left(\frac{1}{2} \cdot (kp + \sqrt{3} - 2k)\right)^2$.

1.6.4 Practice Problems

1. Find V in terms of t by solving the differential equation: $\frac{dV}{dt} = t\sqrt{V}$ if $V\left(\sqrt{2}\right) = 0$.
2. Solve the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$ when $y(1) = 2$.
3. Find a relation between y and x by solving the equation $\frac{dy}{dx} = \cos(x)$ if $y\left(\frac{\pi}{2}\right) = 3$.

4. Solve the differential equation $\frac{dy}{dx} - x = y$ when $y(0) = 4$.
5. Solve the differential equation $\frac{dy}{dx} - x = 2$ when $y(0) = 4$.
6. Solve the differential equation $\frac{dV}{dt} = Vt^2$ when $V(3) = 1$.
7. Solve the differential equation $\frac{dV}{dt} = V^2 t$ when $V(2) = 1$.
8. Solve the differential equation $\frac{dP}{dr} = \frac{P}{r}$ when $P(1) = 2$.

1.6.5 Practice Problems Solutions

1. $V = (t^2 - 2)^2$
2. $y = \sqrt[3]{x^3 + 7}$
3. $y = \sin(x) + 2$
4. Cannot be solved with this technique.
5. $y = \frac{1}{2}x^2 + 2x + C$
6. $V = e^{-9} e^{\frac{1}{3}t^3}$
7. $V = \frac{1}{-\frac{1}{2}t^2 + 3}$
8. $P = 2r$

1.6.6 Homework: Separation of Variables

Problem 1.53 Solve the differential equation $\frac{dJ}{dm} = J$ when $J(2) = 1$.

Problem 1.54 Solve the differential equation $\frac{dL}{dt} = L + 6$ when $L(1) = -5$.

Problem 1.55 Solve the differential equation $\frac{dy}{dx} = \frac{\cos x}{\sin y}$ when $y\left(\frac{\pi}{2}\right) = \pi$.

Problem 1.56 Solve the differential equation $\frac{dy}{dx} = \frac{x}{y}$ when $y(0) = 4$.

Problem 1.57 Solve the differential equation $\frac{dx}{dy} = \frac{\frac{1}{2}y}{\sqrt{x}}$ when $x(2) = 0$.

Problem 1.58 Solve the differential equation $\frac{dy}{dx} - 3y = 0$ when $y(8) = 1$.

1.7 Definite Integrals and the FTC

Objectives — Definite Integrals and the FTC.

- Explain the difference between definite integrals and indefinite integrals.
- Use the Fundamental Theorem of Calculus to evaluate definite integrals.
- Redefine the bounds in a given definite integral when using $u-$substitution.

Vocabulary — Bounds.
Bounds are a new feature we will begin to see in our problems. Quite simply, they tell us from where to where an integral should be done. What is it bounded by? We will see in the next unit exactly what this means geometrically, but for now, you can consider them to be the starting and stopping points of an integral. For an integral like: $\int_a^b f(x)\,dx$, we say the bounds go from a to b. They always start at the bottom and go up.

The bounds are the feature that now make the constant of integration unnecessary. See A Word About the Constant of Integration on page 65 for a little more information.

Vocabulary — Definite Integral.
"Antiderivative" is the technically correct term to use instead of "integral." In calculus, there are two kinds of integrals – the **indefinite integral** (also called the **antiderivative**) and the **definite integral**. We will learn about definite integrals later; in the next few lessons we will focus on antiderivatives. Mathematically, they are closely related, however, the answer to an antiderivative is always an equation, whereas the answer to a definite integral is always a number.

Most people don't bother with (or aren't aware of) the distinction and simply use the term **integral**; however, it is important to note that there is actually a difference.

Property	**Definite Integral** $\int_1^2 x\,\mathrm{d}x = \frac{3}{2}$	**Indefinite Integral** $\int x\,\mathrm{d}x = \frac{1}{2}x^2 + C$
Bounds on the integral symbol	X	
Evaluates to an expression (antiderivative)		X
Evaluates to a number (integral)	X	
Requires a constant of integration		X

Vocabulary — Fundamental Theorem of Calculus.
The Fundamental Theorem of Calculus (FTC) is what connects definite integrals to indefinite integrals – it's what connects antiderivatives to the actual integral.

Theorem 1.7.1 — The Fundamental Theorem of Calculus. Let f be a continuous function on the closed interval $[a,b]$, and let F represent the antiderivative of f. The FTC tells us that

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

In the next unit, you will learn how this works from a conceptual, geometric sense. But for now, we will simply learn the process of evaluating definite integrals.

1.7.1 Fundamental Theorem of Calculus

Example 1.56 Find $\int_1^2 x\,\mathrm{d}x$ ■

Let's refer back to Theorem 1.7.1 which states, $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$. It is important to note that the lowercase $f(x)$ represents the expression we are trying to integrate, while the uppercase $F(x)$ represents the antiderivative of $f(x)$. Let's see this in action:

The antiderivative of x will be $\frac{1}{2}x^2$. We have seen this in previous sections and should be familiar with it. Note that the $+C$ is no longer necessary because we are evaluating a definite integral instead of the indefinite integrals we are used to.

Going back to what we know from Theorem 1.7.1, we should plug the $b-$value in first (in this case it is 2) and the $a-$value second (in this case, it is 1). We'll then subtract these two things:

$$F(b) - F(a) = F(2) - F(1) = \frac{1}{2}(2)^2 - \frac{1}{2}(1)^2 = 2 - \frac{1}{2} = \frac{3}{2}$$

Example 1.57 Find $\int_{-1}^{5}(x^2-2x+5)\,dx$ ■

We will perform the same process, but we'll introduce a new notation here. First, we compute the antiderivative of $(x^2-2x+5)\,dx$ which happens to be $\frac{1}{3}x^3-x^2+5x$. (Remember, we don't need the $+C$ anymore, since this is now a definite integral.)

Because the substitution usually takes a couple of extra steps, we usually write out what we have so far before proceeding. We write it like this:

$$\left(\tfrac{1}{3}x^3-x^2+5x\right)\Big|_{-1}^{5}$$

Notice that we have now written the antiderivative in the parentheses and the bounds are written on the right in the same order they were written before.

This vertical bar is called, "such that." So, the expression we have written above would be said, "$\left(\frac{1}{3}x^3-x^2+5x\right)$ such that x goes from -1 to 5." Notice that the bottom number is said first. It goes from the bottom to the top.

We are ready to evaluate, we obtain:

$$\begin{aligned}
\int_{-1}^{5}&(x^2-2x+5)\,dx\\
&=\left(\frac{1}{3}x^3-x^2+5x\right)\Big|_{-1}^{5}\\
&=\left[\frac{1}{3}(5)^3-(5)^2+5(5)\right]-\left[\frac{1}{3}(-1)^3-(-1)^2+5(-1)\right]\\
&=\left[\frac{125}{3}-25+25\right]-\left[-\frac{1}{3}-1-5\right]\\
&=\frac{125}{3}-\left(-\frac{1}{3}-6\right)\\
&=\frac{125}{3}-\left(-\frac{1}{3}-\frac{18}{3}\right)\\
&=\frac{125}{3}-\left(-\frac{19}{3}\right)\\
&=\frac{125}{3}+\frac{19}{3}=\frac{144}{3}=46
\end{aligned}$$

So, we see that $\int_{-1}^{5}(x^2-2x+5)\,dx=46$, a number just as we anticipated.

Example 1.58 Find $\int_{0}^{\frac{\pi}{2}}\sin x\,dx$ ■

We will fearlessly approach this in the same way:

$$\int_0^{\frac{\pi}{2}} \sin x \, dx$$
$$= (-\cos x)\Big|_0^{\frac{\pi}{2}}$$
$$= \left[-\cos\left(\frac{\pi}{2}\right)\right] - [-\cos(0)]$$
$$= (0) - (-1) = 1$$

We conclude that $\int_0^{\frac{\pi}{2}} \sin x \, dx = 1$.

Example 1.59 Find $\displaystyle\int_0^{\pi} (x^2 - 3x + \cos x) \, dx$ ■

$$\int_0^{\pi} (x^2 - 3x + \cos x) \, dx$$
$$= \left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + \sin x\right)\Big|_0^{\pi}$$
$$= \left[\frac{1}{3}(\pi)^3 - \frac{3}{2}(\pi)^2 + \sin(\pi)\right] - \left[\frac{1}{3}(0)^3 - \frac{3}{2}(0)^2 + \sin(0)\right]$$
$$= \left(\frac{\pi^3}{3} - \frac{3\pi^2}{2} + 0\right) - (0)$$

So, $\int_0^{\pi} (x^2 - 3x + \cos x) \, dx = \frac{\pi^3}{3} - \frac{3\pi^2}{2}$. There is nothing to motivate me to simplify this further.

1.7.2 Using the FTC with u-Substitution

We are now at a point where we can use all the skills recently learned – integration, $u-$substitution, and the FTC. There is only one catch with which we must be careful.

Example 1.60 Find $\displaystyle\int_{-3}^{4} (3x+4)^5 \, dx$ ■

We should first notice that this problem will require that we use $u-$substitution:

$$u = 3x + 4$$
$$du = 3 \, dx$$
$$\tfrac{1}{3} \, du = dx$$

This will give us: $\frac{1}{3}\displaystyle\int_{-3}^{4} u^5 \, du$.

However, this is WRONG! This is a common mistake that countless Calculus students make everyday.

Notice that using the terminology from Example 1.57, we would say, "as x goes from -3 to 4. These bounds are specifically for the $x-$value. They do not apply to u. Since we have a new variable, we will need new bounds. There are two ways we can handle this:

Option 1 – Changing the bounds
We must ask ourselves: What will the $u-$value be when $x = -3$? And correspondingly, what will the $u-$value be when $x = 4$?

We can plug these into the expression we obtained earlier for u and express this as: $u(-3) = 3(-3)+4 = -5$ and $u(4) = 3(4)+4 = 16$. Therefore, our integral would be correctly expressed as:

$$\frac{1}{3}\int_{-5}^{16} u^5 \, \mathrm{d}u$$

The same process can now be followed to finish evaluating the integral.

$$\begin{aligned}
&\frac{1}{3}\int_{-5}^{16} u^5 \, \mathrm{d}u \\
&\qquad = \frac{1}{3}\left(\frac{1}{6}u^6\right)\Bigg|_{-5}^{16} \\
&\qquad = \frac{1}{3}\left\{\left[\frac{1}{6}(16)^6\right] - \left[\frac{1}{6}(-5)^6\right]\right\} \\
&\qquad = \frac{1}{18}\left(16^6 - 5^6\right)
\end{aligned}$$

There is no reason to simplify further, we obtain our answer: $\int_{-3}^{4}(3x+4)^5 \, \mathrm{d}x = \frac{1}{18}\left(16^6 - 5^6\right)$.

Option 2 – Using an indefinite integral
Alternatively, we can leave the bounds alone, treat it as an indefinite integral temporarily, then plug back in once we have finished. This means we would have:

$$\begin{aligned}
&\frac{1}{3}\int u^5 \, \mathrm{d}u \\
&\qquad = \frac{1}{3}\left(\frac{1}{6}u^6\right)
\end{aligned}$$

Since $u = 3x+4$, we actually have: $\frac{1}{3} \cdot \frac{1}{6}(3x+4)^6$. That means:

$$\begin{aligned}
&\int_{-3}^{4}(3x+4)^5 \, \mathrm{d}x \\
&\qquad = \frac{1}{3} \cdot \frac{1}{6}(3x+4)^6\Big|_{-3}^{4} \\
&\qquad = \frac{1}{18}\left\{[3(4)+4]^6 - [3(-3)+4]^6\right\} \\
&\qquad = \frac{1}{18}\left(16^6 - (-5)^6\right) \\
&\qquad = \frac{1}{18}\left(16^6 - 5^6\right)
\end{aligned}$$

We see that we have obtained the same solution we obtained using Option 1. Therefore, either option works fine.

Theorem 1.7.2 — Definite Integrals and $u-$Substitution. When working with definite integrals involving $u-$substitution, we must always remember to deal with the bounds.

Option 1: The bounds can be changed to reflect the variable being worked with. Then there is no need to replace the $u-$terms with the original expressions.
Option 2: The integral can be temporarily performed as an indefinite integral. Once a solution has been obtained, the original expressions can replace the $u-$terms, allowing the bounds to remain unchanged.

Example 1.61 Find $\int_{-\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} 2x\cos\left(x^2\right)\mathrm{d}x$ ■

For the sake of further illustration, we will offer both options on this example.

Option 1 – Changing the bounds
We choose:

$$u = x^2 \qquad\qquad \mathrm{d}u = 2x\,\mathrm{d}x$$

We solve for the new bounds:

$$u\left(-\sqrt{\frac{\pi}{2}}\right) = \left(-\sqrt{\frac{\pi}{2}}\right)^2 = \frac{\pi}{2} \qquad\qquad u\left(\sqrt{\pi}\right) = \left(\sqrt{\pi}\right)^2 = \pi$$

We are now able to evaluate:

$$\int_{-\frac{\pi}{2}}^{\pi} \cos u\,\mathrm{d}u$$

$$= \sin u\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= \sin(\pi) - \sin\left(\frac{\pi}{2}\right)$$

$$= 0 - 1 = -1$$

Option 2 – Using an indefinite integral
Again, we choose:

$$u = x^2 \qquad\qquad \mathrm{d}u = 2x\,\mathrm{d}x$$

We now evaluate as an indefinite integral:

$$\int \cos u\,\mathrm{d}u = \sin u$$

Substituting back in for the $u-$term and bringing our bounds for x back, we now have:

$$= \sin\left(x^2\right)\Big|_{-\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}}$$

$$= \left[\sin\left(\sqrt{\pi}\right)^2\right] - \left[\sin\left(\sqrt{\frac{\pi}{2}}\right)^2\right]$$

$$= [\sin(\pi)] - \left[\sin\left(\frac{\pi}{2}\right)\right]$$

$$= 0 - 1 = -1$$

Regardless of which way we choose to approach this integral, we obtain the solution: $\int_{-\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} 2x\cos\left(x^2\right)\mathrm{d}x = -1$.

Example 1.62 Find $\int_{-5}^{10} \left(\frac{1}{6x+4} - x^3\right) \, dx$ ■

This problem will require us to split up the integral. Notice that the first term will need $u-$substitution, while the last two will not. Because of the issue with the bounds, it is imperative to split these up.

$$\int_{-5}^{10} \left(\frac{1}{6x+4} - x^3\right) \, dx = \int_{-5}^{10} \left(\frac{1}{6x+4}\right) \, dx - \int_{-5}^{10} x^3 \, dx$$

We will use only option 1 for this particular example, though either option is acceptable. Choose:

$$u = 6x+4 \qquad du = 6\,dx \qquad \frac{1}{6}\,du = dx$$

Giving us new bounds:

$$u(-5) = 6(-5)+4 = -26 \qquad u(10) = 6(10)+4 = 64$$

We remember to change the bounds of only the first integral as that is the only one that has changed variables to u.

$$\frac{1}{6}\int_{-26}^{64} \frac{1}{u}\,du - \int_{-5}^{10} x^3\,dx$$

$$= \frac{1}{6}\ln|u|\Big|_{-26}^{64} - \frac{1}{4}x^4\Big|_{-5}^{10}$$

$$= \frac{1}{6}\left[\ln|64| - \ln|-26|\right] - \frac{1}{4}\left[(10)^4 - (-5)^4\right]$$

$$= \frac{1}{6}\left[\ln(64) - \ln(26)\right] - \frac{1}{4}\left(10^4 - 5^4\right)$$

This gives us our final answer: $\int_{-5}^{10} \left(\frac{1}{6x+4} - x^3\right) \, dx = \frac{1}{6}\left[\ln(64) - \ln(26)\right] - \frac{1}{4}\left(10^4 - 5^4\right)$.

Example 1.63 Find $\int_{0}^{\frac{\pi}{2}} \sin(x) e^{\cos(x)} \, dx$ ■

We will use option 1 here as it is usually faster, again, either option is acceptable.

$$u = \cos(x) \qquad du = \sin(x)\,dx$$

$$u(0) = \cos(0) = 1 \qquad u\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$

Notice that we will keep the bounds in the order they began. The bottom was bounded by 0, so the bottom will continue to be bounded by the corresponding number, 1 and vice versa, even though they are now out of order.

$$\int_{1}^{0} e^u\,du$$

$$= (e^u)\Big|_1^0$$

$$= e^0 - e^1$$

$$= 1 - e$$

$$\int_{0}^{\frac{\pi}{2}} \sin(x) e^{\cos(x)}\,dx = 1 - e.$$

AP The makers of the AP test will test you to see if you understand this concept of dealing with the bounds. There is usually at least one question that will ask you to determine what the new bounds will be after a given u-substitution is performed. Be sure that you are equally comfortable with either option.

1.7.3 Switching the Bounds

Convention dictates that we list the bounds with the smaller number on bottom and the larger number on top. However, mathematically, this doesn't have to be the case and in the last example, we even saw how this might work. We have two choices when the bounds are in the reverse order: we can either leave them alone or we can flip them. Let's look at an example of each.

Example 1.64 Find $\displaystyle\int_3^0 x^4\,dx$ by leaving the bounds alone. ■

We will simply solve this the way we normally would, ignoring the nagging feeling that the bounds are amiss:

$$\int_3^0 x^4\,dx$$
$$= \frac{1}{5}x^5\Big|_3^0$$
$$= \frac{1}{5}(0)^5 - \frac{1}{5}(3)^5$$
$$= -\frac{3^5}{5}$$

Let's keep this answer in mind for the next example: $\int_3^0 x^4\,dx = -\frac{3^5}{5}$.

Example 1.65 Find $\displaystyle\int_3^0 x^4\,dx$ by switching the bounds. ■

Let's just do what it says and see what happens. We will switch the bounds and obtain:

$$\int_0^3 x^4\,dx$$
$$= \frac{1}{5}x^5\Big|_0^3$$
$$= \frac{1}{5}(3)^5 - \frac{1}{5}(0)^5$$
$$= \frac{3^5}{5}$$

The correct answer, as we saw in Example 1.64 was $-\frac{3^5}{5}$, however, here we got $\frac{3^5}{5}$, which means it is almost correct, it is off by a negative.

This will always be the case when we are switching bounds. If we switch the bounds, we should also switch the sign of our answer.

Theorem 1.7.3 — Switching the Bounds. The bounds of any integral can be switched by negating the integral: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Example 1.66 Find $\int_0^{-\frac{\pi}{2}} \sin(x)\,dx$ by switching the bounds. ■

We will do as we have said and switch the bounds, then make our answer negative.

$$\int_0^{-\frac{\pi}{2}} \sin(x)\,dx$$

$$= -\int_{-\frac{\pi}{2}}^{0} \sin(x)\,dx$$
$$= -(-\cos(x))\Big|_{-\frac{\pi}{2}}^{0}$$
$$= [\cos(0)] - \left[\cos\left(-\frac{\pi}{2}\right)\right]$$
$$= 1 - 0$$

Our final answer is $\int_0^{-\frac{\pi}{2}} \sin(x)\,dx = 1$

Once again, we see that it would have been possible to use either of the two options presented: leave the bounds alone and integrate as normal, or switch the bounds and negate the integral. One might wonder why we would ever engage in the second option, as it is certainly not the most direct way of evaluating the integral. We will see in Section 1.7.4 how this trick can be incredibly useful.

1.7.4 Working with Abstract Problems

Sometimes, we are given problems that test our understanding of the material and not just our ability to go through the steps. We refer to these as "abstract" problems because they feature concepts, not actual expressions. We should look at an example to get a better idea of what this means.

Example 1.67 If $\int_1^3 f(x)\,dx = 5$ and $\int_1^5 f(x)\,dx = 3$, what is $\int_3^5 f(x)\,dx$? ■

This is actually a fairly straight-forward problem because we only have one function, $f(x)$, and no $u-$substitution. The problems will get more challenging as we go.

In this case, the integral of the function from 1 to 3 is 5, and from 1 to 5 is 3. Since we only want from 3 to 5, we will subtract, $3-5=-2$.

So we see that $\int_3^5 f(x)\,dx = -2$.

Example 1.68 Let $f(x)$ be a function such that $\int_{13}^{85} f(x)\,dx = 3$. Find $\int_1^{10} f(8x+5)\,dx$. ■

Hopefully, it is apparent that this problem will require the use of $u-$substitution. We must remember as we go, that since this is a definite integral and we are using $u-$substitution, we will need to change the bounds.

$$u = 8x + 5 \qquad \mathrm{d}u = 8\,\mathrm{d}x \qquad \frac{1}{8}\,\mathrm{d}u = \mathrm{d}x$$

$$u(1) = 8(1) + 5 = 13 \qquad u(10) = 8(10) + 5 = 85$$

We see now that we have conveniently obtained the bounds from the given integral (that happened on purpose). We can use that now. So, we have this:

$$\int_1^{10} f(8x+5)\,\mathrm{d}x = \tfrac{1}{8}\int_{13}^{85} f(u)\,\mathrm{d}u$$

Since we know that $\int_{13}^{85} f(x)\,\mathrm{d}x = 3$, we can swap that out. The fact that we are using an x here and what we're swapping it out has a u doesn't matter. As long as it's $f(\mathit{something})\,\mathrm{d}\mathit{something}$, then that is 3. The letter used is arbitrary.

$$\int_1^{10} f(8x+5)\,\mathrm{d}x = \tfrac{1}{8}\int_{13}^{85} f(u)\,\mathrm{d}u = \tfrac{1}{8}\cdot 3$$

And so we have our answer: $\int_1^{10} f(8x+5)\,\mathrm{d}x = \frac{3}{8}$.

Example 1.69 Let $f(x)$ be a function such that $\displaystyle\int_{-2}^{3} f(x)\,\mathrm{d}x = 3.1$ and $\displaystyle\int_{3}^{7} f(x)\,\mathrm{d}x = -2$. Find $\displaystyle\int_{-2}^{7} (-1 + 3f(x))\,\mathrm{d}x$ ■

It is easiest if we split this integral up:

$$-1\int_{-2}^{7} \mathrm{d}x + 3\int_{-2}^{7} f(x)\,\mathrm{d}x$$

This is now much easier to approach. Note that:

$$\int_{-2}^{7} f(x)\,\mathrm{d}x = \int_{-2}^{3} f(x)\,\mathrm{d}x + \int_{3}^{7} f(x)\,\mathrm{d}x = 3.1 + -2 = 1.1$$

Let us now continue with the actual computations:

$$\begin{aligned} -1\int_{-2}^{7} \mathrm{d}x + 3\int_{-2}^{7} f(x)\,\mathrm{d}x & \\ &= -1x\Big|_{-2}^{7} + 3\int_{-2}^{7} f(x)\,\mathrm{d}x \\ &= -(7) - (-(-2)) + 3(1.1) = -7 - 2 + 3.3 = -5.7 \end{aligned}$$

We can confidently say that $\displaystyle\int_{-2}^{7} (-1 + 3f(x))\,\mathrm{d}x = -5.7$.

Example 1.70 Assume that $\displaystyle\int_a^b e^u\,\mathrm{d}x = 4$ for some $a, b \in \mathbf{R}$. What is $\displaystyle\int_{a^2}^{b^2} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$? ■

This will require us to use $u-$substitution. We can see that we should use:

$$u=\sqrt{x} \qquad \mathrm{d}u=\frac{1}{2}x^{-\frac{1}{2}}\,\mathrm{d}x \qquad 2\,\mathrm{d}u=x^{-\frac{1}{2}}\,\mathrm{d}x$$

$$u(a^2)=\sqrt{a^2}=a \qquad u(b^2)=\sqrt{b^2}=b$$

Substituting all of this in, we have:

$$2\int_a^b e^u\,\mathrm{d}u$$

$$=2(4)=8$$

So, our final answer is: $\int_{a^2}^{b^2}\frac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x=8$.

Example 1.71 Consider a function f such that $\int_{-1}^{1} f(u)\,\mathrm{d}u=3$. Use this to evaluate $\int_0^{\pi}\sin x\cdot f(\cos x)\,\mathrm{d}x$ ■

Let's begin by using $u-$substitution. Remember, when we use $u-$substitution, we're always trying to pick out the argument of the function. In this case, it's actually quite easy because the argument is blatantly there:

$$u=\cos x \qquad \mathrm{d}u=-\sin x\,\mathrm{d}x \qquad -\mathrm{d}u=\sin x\,\mathrm{d}x$$

$$u(\pi)=\cos(\pi)=-1 \qquad u(0)=\cos(0)=1$$

We can now work on evaluating:

$$\int_0^{\pi}\sin x\cdot f(\cos x)\,\mathrm{d}x$$

$$=-\int_1^{-1} f(u)\,\mathrm{d}u$$

$$=\int_{-1}^{1} f(u)\,\mathrm{d}u$$

$$=3$$

So, this ended up actually being quite straight-forward, we conclude that $\int_0^{\pi}\sin x\cdot f(\cos x)\,\mathrm{d}x=3$.

1.7.5 Practice Problems

1. Find $\int_{-5}^{7}(3x^2-2x)\,\mathrm{d}x$.
2. Find $\int_0^{\pi}\sec^2 x\,\mathrm{d}x$.
3. Find $\int_2^3(x-4)(x^2-8x)^5\,\mathrm{d}x$.
4. Find $\int_{-4}^{4}(2x+2)e^{x^2+2x}\,\mathrm{d}x$.
5. Find $\int_3^5\frac{1}{x}\,\mathrm{d}x$.
6. Find $\int_a^b\frac{\cos\ln(x)}{x}\,\mathrm{d}x$.
7. Find $\int_{-3}^{0}e^{2x}(e^{2x}+1)^3\,\mathrm{d}x$.
8. Let $\int_0^1 f(x)\,\mathrm{d}x=4$, $\int_1^{10} f(x)\,\mathrm{d}x=-3$, and $\int_{10}^{11} f(x)\,\mathrm{d}x=2$. Find $\int_0^{11} f(x)\,\mathrm{d}x$.

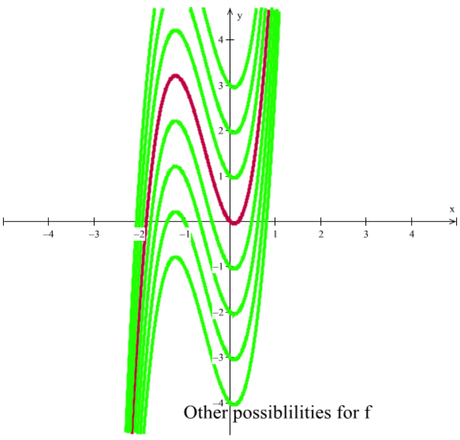
9. If $\int_1^5 f(x)\,dx = -1$ and $\int_1^{15} f(x)\,dx = 3$, what is $\int_2^3 x \cdot f(2x^2-3)\,dx$?
10. Consider a function f such that $\int_1^3 f(x)\,dx = 6$. Evaluate $\int_1^2 [4x + f(2x-1)]\,dx$.
11. When $\int_a^b \ln u\,du = 2$, then $\int_{\sqrt{a}}^{\sqrt{b}} \ln(x^2) \cdot x\,dx =$
12. Find $\int_2^{-5} 3x^2\,dx$ by leaving the bounds alone.
13. Find $\int_{-1}^{-2} (4x^3+2)\,dx$ by switching the bounds.

1.7.6 Practice Problems Solutions

1. $7^3 - 7^2 + 5^3 - 5^2$
2. 0
3. $\frac{1}{12}\left(15^6 - 12^6\right)$
4. $e^{24} - e^8$
5. $\ln 5 - \ln 3$
6. $\sin(\ln|b|) - \sin(\ln|a|)$
7. $\frac{1}{4}\left(2^4 - \left(e^{-6}+1\right)^4\right)$
8. 3
9. 1
10. 9
11. 1
12. -133
13. 13

1.7.7 A Word About the Constant of Integration

If we remember back to the Word About the Constant of Integration on page 11 and page 40, we recall that the constant of integration was necessary when we didn't know exactly where the integral would be based on the antiderivative alone and needed a point to determine which curve it was.



Because we now have bounds, the location of the function is not important. We are only concerned with the shape of the function. This is why the $+C$ is no longer necessary. This argument will be laid out geometrically in the corresponding Word About the Constant of Integration later on page **??**.

1.7.8 Homework: Definite Integrals and the FTC

Problem 1.59 Find $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\,dx$.

Problem 1.60 Find $\int_0^1 (x^3 - x^2)\,dx$.

Problem 1.61 Find $\int_1^3 x(x^2+2)\,dx$.

Problem 1.62 Find $\int_{-2}^1 (2x+3)(2x^2-1)\,dx$.

Problem 1.63 Find $\int_{-1}^1 \left((3x+4)^8 + x^2\right)\,dx$.

Problem 1.64 Find $\int_7^{10} f(x)\,dx$ if $\int_1^{10} f(x)\,dx = 3$ and $\int_1^7 f(x)\,dx = -1$.

Problem 1.65 Find $\int_0^\pi (\sin(3x) + x)\,dx$.

Problem 1.66 Find $\int_1^5 x^2(x^3+9)\,dx$.

Problem 1.67 Assume $\int_2^4 \sin u\,du = a$, for some $a \in \mathbf{R}$. Find $\int_4^{16} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ in terms of a.

Problem 1.68 Find $\int_b^f (4x + e^x)\,dx$.

Problem 1.69 Find $\int_1^2 [2x - 3f(3x-1)]\,dx$ if $\int_2^5 f(u)\,du = 10$.

Problem 1.70 Find $\int_3^1 xe^{x^2}\,dx$.

Problem 1.71 Which of the following is equal to: $\int_3^6 e^{5x+4}\,dx$?

a $\frac{1}{5}\int_3^6 e^u\,du$

b $\frac{1}{5}\int_{19}^{34} e^u\,du$

c $\int_{19}^{34} e^{5x+4}\,dx$

d $\int_6^3 e^{5x+4}\,dx$

Problem 1.72 Which of the following is equal to: $\int_\pi^{2\pi} x\sin x^2\,dx$?

a $\int_{\pi^2}^{4\pi^2} \frac{1}{2}\sin(u)\,du$

b $\int_{\pi^2}^{4\pi^2} 2x\sin(x^2)\,du$

c $\int_\pi^{2\pi} \sin(u)\,du$

d $\int_{\pi^2}^{4\pi^2} \sin(u)\,du$

1.8 Integral Applications

As you learn about how integration works, it's important to understand what we use the operation for. In this section, we'll investigate some scenarios where an integral would be useful and learn how to set up the integrals.

Objectives — Integral Applications.

- Integrate functions to determine the amount of something based on its rate.
- Determine when the initial condition is relevant to a given problem.

Let's begin by recalling the direction of functions: $f \to f' \to f''$.

We can adapt this to a more conceptual understanding. The first derivative, f' can represent the rate of change for something, while f would represent the actual quantity of that something. In this section, we will not be working with the second derivative at all. So we now have something like this:

f	$\leftrightarrow$	Amount
f'	$\leftrightarrow$	Rate
f''	$\leftrightarrow$	*Not used today*

For example:

f' (Rate)	f (Amount)
The population growth rate of a penguin colony.	The actual number of penguins at a given time.
The rate of cars passing through an intersection.	The number of cars that passed through the intersection.
The rate of people entering a zoo.	The number of people who entered the zoo.
The rate of water leaking from a broken hose.	The amount of water that has leaked from the hose.

There are two types of questions they are likely to ask you on the AP test. The first is the amount of something during a given interval. The second is the amount at a certain point. Let's take for example an example where $L(t)$ represents the rate of water leaking from a faucet. They might ask you how much water leaked only between 2PM and 3PM, or they might ask you how much total water had leaked by 5PM. We will look at each case separately.

1.8.1 Working with Intervals

Let's consider the example with leaking water mentioned above.

Example 1.72 Water is leaking from a broken faucet for 12 hours. The function $L(t) = \sin(t^3 - 3t^2) + 4t^3$ represents the rate at which the water in mL/hour is leaking in the interval $2 \leq t \leq 12$. How much water leaks from the faucet between $t = 2$ and $t = 3$? ■

Note that we will compare this question with Example 1.75 soon.

We are given the rate: $L(t)$ and we wish to have the amount. Recall that means we are going from the first derivative to the original function. We must integrate. We will use bounds for this integral.

Notice that there are two intervals given, $2 \leq t \leq 12$ and $2 \leq t \leq 3$. The first interval only tells us where the given equation is allowed to work. We're not so concerned with that. The second interval is what they are asking us to investigate, so that is the interval we will use as the bounds on our integral.

Our integral will be: $\int_2^3 L(t)\,dt$. Writing the $L(t)$ instead of the entire function is acceptable since $L(t)$ has already been defined for us. However, we would certainly use the entire function when evaluating.

Oftentimes, these problems ask us to get an approximation as opposed to the exact answer we have highlighted above. This would be an exceptionally difficult integral to do by hand, so you should use a calculator. By doing this, we obtain: 64.954 or 64.955 mL of water that leaked from the faucet in that hour.

Example 1.73 The birth rate of porcupines per year in a Calculus Land is given by the function $B(t) = 500\ln(4t^3 - 2t) + \cos(3t)$ which is defined for the interval $2 \leq t \leq 10$. How many porcupines were born in this interval, rounded to the nearest porcupine? ■

Here, they are telling us to use the given interval, $(2,10)$, there is no need to dwell on it. Therefore, we have: $\int_2^{10} \left(500\ln(4x^3 - 2x) + \cos(3x)\right) dt \approx 25,901.600$. Rounded to the nearest whole number, we have 25,902 porcupines.

AP The AP test is not as concerned with the approximation you obtain, but more with the approach you took. If you simply write the numerical answer and not the integral used, you risk receiving no points. Therefore, from this point forward, we will highlight the entire answer that you should offer on the free response questions.

Example 1.74 The rate of students per month applying to Defenestration University during the first 6 months applications are accepted can be predicted from the function $A(t) = (0.3t-3)^3 + 30(0.3t-3)^2 - \frac{1}{0.3t-3}$. How many students are expected to apply from $t = 2$ to $t = 6$? ■

Acceptable AP test free response answer: $\int_2^6 A(t)\,\mathrm{d}t \approx 379$ or 380 students.

1.8.2 Working with Initial Conditions

In all of the examples we looked at above, we only cared about what was happening in a specific interval. It didn't matter what was happening before that interval. Now, we will look at cases where the initial condition is important to the problem. Let's consider the example we started with before, but with the question adjusted.

Example 1.75 Water is leaking from a broken faucet for 12 hours. The function $L(t) = \sin(t^3 - 3t^2) + 4t^3$ represents the rate at which the water in mL is leaking in the interval $2 \leq t \leq 12$. By $t = 2$, 10 mL of water had already leaked from the faucet. How much water has leaked from the faucet by $t = 3$? ■

Notice how this question differs from that in Example 1.72. In that example, they ask us how much water leaks *only during that interval*. In this example, they ask us how much water has leaked *by a certain time*.

The set up in Example 1.72 did not require an initial condition. It simply didn't matter how much water had already looked. In this example, however, we must take that extra water that already leaked into consideration, so this problem will require an initial condition.

Our answer will look like this: $10 + \int_2^3 L(t)\,\mathrm{d}t \approx 74.954$ or 75.955 mL.

Notice that this answer is 10 higher than the answer in Example 1.72. This makes sense as the bounds are the same, the only difference was the added 10 in front.

Let's discuss the structure of the expression we set up.

The basic idea is that we want to find how much water leaked in a certain period of time (in this case from $t = 2$ to $t = 3$) and then add on the amount that we already knew about. Notice that the bottom bound is a pair with the quantity being added, i.e. at $t = 2$ (the bottom bound) we have an initial amount of 10 (the quantity being added). This will always be the case for these types of problems, and can actually be quite formulaic. In fact, we can formalize this:

Theorem 1.8.1 — Incorporating Initial Conditions. When involving an initial condition in an integral application, we can use the form:

$$\text{initial amount} + \int_{\text{initial time}}^{\text{desired time}} f(t)\,\mathrm{d}t$$

Consider: The integral portion of the expression above represents the amount by which something has changed during a specific period. If we want to know how much of something there is, including what we started with, we would add the initial condition to the amount it changed by.

Example 1.76 Ariel, a world famous lobster hunter, is able to catch lobsters at a rate given by $L(t) = 4t^3 - 2t$, where t is measured in hours and $L(t)$ is measured in lobsters per hour and the function is valid during the course of her work day which is 8 hours long. When she arrives at her boat, she discovers there are already 47 lobsters in her traps. After 3 hours of intense hunting, how many would she have caught, including the lobsters she started the day with? ■

Here again, we have an initial condition. We want to know how many total she has caught by the end of 3 hours, so we have to consider how many she catches during those 3 hours (which will be the integral part of our expression) and then add the number of lobsters she started with (the initial condition of 47 lobsters).

Using the ideas from 1.8.1, we would have: initial lobsters $+ \int_{\text{initial time}}^{\text{desired time}} L(t)\,\mathrm{d}t$.

We should now plug everything in and integrate. Because the function $L(t)$ is fairly straightforward to integrate, we would expect to find this on the non-calculator section of the test and would be expected to integrate it by hand.

$$47 + \int_0^3 \left(4t^3 - 2t\right)\,\mathrm{d}t$$

$$= 47 + \left(t^4 - t^2\right)\Big|_0^3$$
$$= 47 + \left(3^4 - 3^2\right) - \left(0^4 - 0^2\right) = 47 + 81 - 9 = 119$$

The answer on the free response part of the AP test could be written as: $47 + \int_0^3 \left(4t^3 - 2t\right)\,\mathrm{d}t = 119$ lobsters.

Example 1.77 A software company develops code that can determine the decimals of an irrational number at a rate of $r(t) = 4t^3 - \cos t$ as measured in decimals per second when $0 \le t \le 15$. If the first 34 decimal digits of this number are already known, how many decimals would be known after they run the program for 2 hours? ■

We'll be starting this expression with our initial condition of 34 decimal digits and then adding the new hamburgers that come to us through the integral. We will have:

$$34 + \int_0^2 \left(4t^3 - \cos t\right)\,\mathrm{d}t$$

$$= 34 + \left(t^4 - \sin t\right)\Big|_0^2$$
$$= 34 + \left(2^4 - \sin(2)\right) - \left(0^4 - \sin(0)\right)$$
$$= 34 + 16 - \sin(2) = 50 - \sin(2) \approx 49.091$$

The final answer would look like this: $34 + \int_0^2 \left(4t^3 - \cos t\right)\,\mathrm{d}t \approx 49$ decimal digits.

1.8.3 Working with Multiple Functions

Though many scenarios require only one function, there are circumstances where a rate is broken into multiple functions, some of which may also involve an initial condition. Let's consider an example:

Example 1.78 Scientists studying a Snurgle population have established a birth rate governed by the function $B(t) = 4t^3 - \ln(3t^3)$ and a death rate governed by the function $D(t) = 3\sin(t^2) + 1$. Both of these functions are found to be valid from $t = 0$ to $t = 12$ where t represents the number of months since the Snurgle study first began. By how much does the Snurgle population change in the first 6 months of the study?

N.B. I am a Mathematician, not a Biologist. Questions regarding the taxonomy or zoology of a Snurgle will not be addressed in this book and are left as an exercise for the reader. ■

The overall population rate of the Snurgles would of course require us to subtract the death rate from the birth rate, that is, our population rate is $B(t) - D(t)$. Since we are looking for the amount during a specific interval, we're not concerned with an initial condition. Therefore, we will have the answer: $\int_0^6 (B(t) - D(t))\, dt \approx 1267.24$.

Note that when this is input into the calculator, there should be parentheses around the function $D(t)$, otherwise there will be a sign error. It may alternatively be calculated: $\int_0^6 B(t)\, dt - \int_0^6 R(t)\, dt$.

Example 1.79 Jesse has discovered that, for $0 \leq t \leq 365$, the growth of his bank account can be modeled by the function $G(t)$ and his expenses can be modeled by the function $E(t)$ (where t is measured in days, and $G(t)$ and $E(t)$ are measured in dollars per day). If he has \$483.27 on Day 1, write an expression to represent how much money is in his bank account by Day 87. ■

In this example, we have an initial condition, the \$483.27. The question is: do we need it? The answer is yes. How can we know how much money is in his account at the end, if we don't know what he started with? The integral portion of our expression only represents the change of money. We must know what we started with to know what that change contributed to.

We go back to our Theorem 1.8.1 and find that we should be writing: $483.27 + \int_1^8 (G(t) - E(t))\, dt$.

Example 1.80 Alfonso and Sarah are both TAs (teaching assistants) for a professor at MIT. They have many papers to grade and have decided to determine functions that describe the rate at which they can grade during the time frame of one week. Alfonso grades at a rate of $A(t)$ and Sarah grades at a rate of $S(t)$, both are measured in papers per hour. Two days into a grading binge, they have graded 34 papers. How many papers will they grade together over the next 5 days? ■

This problem does give us an initial condition, but is it necessary? We are only being asked how many papers they will grade from Day 2 to Day 5. We're not being asked how many total papers they will have graded by Day 5. So, we are only concerned with the integral, we don't need to worry about an initial conditions. The initial condition is extraneous information.

Another difference from the previous two examples to note is that we will be adding these two functions instead of subtracting them. Alfonso and Sarah are both grading together and their efforts are contributing to a total number of graded papers. Unless they're doing it completely wrong, neither of them should be subtracted.

That gives us an answer of: $\int_2^5 (A(t)+S(t))\,\mathrm{d}t$.

AP On the AP test, they will not tell you if you need an initial condition, or if more than one function should appear in your integral, you must determine these using your reasoning.

1.8.4 Practice Problems

1. Oil is pumped into a refinery at a rate of $P(t)$ while the refined product is pumped out of the refinery at a rate of $R(t)$. Both of these functions are measured as gallons per hour and are defined on the interval $4 \leq t \leq 12$ where t represents the hours of a day. If there are $14,000$ gallons in the refinery at $t=4$, write, but do not evaluate an integral expression which represents the number of gallons that are in the refinery at $t=12$.

2. The growth rate of mold in a petri dish is given by $G(t)$, while the death rate is given by $D(t)$ for the interval $10 \leq t \leq 14$, where both $G(t)$ and $D(t)$ are measured in mL per hour and t is given in hours. At $t=10$ there are 2.8 mL of mold in the petri dish. Write, but do not evaluate, an integral expression representing how much mold grows in the first 12 hours.

3. Ms. Mackay has been very busy and has not had a chance to touch her coffee. Unfortunately, the Second Law of Thermodynamics has set in and the coffee has quickly begun to cool. The coffee cools at a rate given by $C(t) = -21.4\cdot(0.7)^t$ for $0 \leq t \leq 12$. If her coffee is 100° at $t=0$, how hot is it by $t=25$? *Round your answer to the nearest thousandth place, use a calculator.*

4. The rate of snow falling during a violent blizzard can be given by the function $S(t) = \sin(t+5)+4\cos t+10$ for the duration of the blizzard which lasts for 48 hours. Let t represent hours and $S(t)$ be measured by inches per hour. How much snow falls from $t=4$ to $t=15$? *Round your answer to the nearest thousandth place, use a calculator.*

5. Cheyenne is late to work and trying to get there as quickly as she can. She is trying to maintain a velocity of $v(t)=3t^3$ for her 30 minute drive, but is having trouble because the friction of the road and wind pushing against her are slowing her down at a rate of $s(t)=\sec^2 t$. How much distance does she cover from time $t=0$ to time $t=\frac{\pi}{4}$? *Simplify your answer to a whole number, no calculator.*

6. The Infinite Hotel has a rate of new patrons of $n(t) = 4\sqrt{t}+8t$ and a rate of patrons checking out of $c(t)=2t-5+4\sqrt{t}$ over the course of the 3 month peak period. If there are 831 patrons in the hotel at $t=0$ days, how many patrons would there be in the hotel at $t=4$ days? *Simplify your answer to a whole number, no calculator.*

7. The amount of sleep a high school student gets per day, can be modeled by three functions, $P(t)$, $H(t)$, and $V(t)$. $P(t)$ represents parental pressure to go to bed at a reasonable hour, thus adding to the amount of sleep a student gets per night. $H(t)$ represents the loss of sleep due to homework assignments. $V(t)$ represents sleep time lost to video games. Write, but do not evaluate, an integral expression representing the amount of sleep a student gets over the course of one week.

1.8.5 Practice Problems Solutions

1. $14,000+\int_4^{12}(P(t)-R(t))\,\mathrm{d}t$
2. $\int_0^{12}B(t)\,\mathrm{d}t$
3. $100+\int_0^{25}C(t)\,\mathrm{d}t\approx 40.009^\circ$
4. $\int_4^{15}S(t)\,\mathrm{d}t\approx 114.309$ inches
5. $\int_0^{\frac{\pi}{4}}\left(3t^3-\sec^2 t\right)\,\mathrm{d}t=\frac{\pi}{4}-1$
6. $831+\int_0^4(n(t)-c(t))\,\mathrm{d}t=899$ patrons
7. $\int_0^7(P(t)-H(t)-V(t))\,\mathrm{d}t$

1.8.6 Homework: Integral Applications

Problem 1.73 A rancher has recently invested in a granary and finds that they can measure the rate at which grain is used by the function $E(t)=4+3\sqrt{t}$ where t is measured in days and $E(t)$ is measured in bushels per day, for $0\le t\le 31$. How many bushels would the rancher use beginning at Day 4 until Day 9? *Simplify your answer to an integer, no calculator.*

Problem 1.74 A major fast food restaurant assembles hamburgers at a rate given by the function $h(t)=100(6t-2t^2)$ measured in hamburgers per hour during the lunch rush. Before the lunch rush, the restaurant has 12 hamburgers prepared. How many hamburgers will they have assembled by the end of the lunch rush, 3 hours later? *Simplify your answer to an integer, no calculator.*

Problem 1.75 The weight-gain of a baby gremlin during the first 3 weeks of life can be given by the function $g(t)=\sin^2 t-\sin t+80$, which is measured in pounds per day. Splorgy, a newborn gremlin weighs 85 pounds when he is born. How much does his weight change in the first 5 days? *Round your answer to the nearest thousandth place, use a calculator.*

Problem 1.76 Painters, Daryl and Theresa, have been hired to paint a new house. During the week that they have to paint the house, Daryl can paint at a rate of $D(t)=4t^7-2t^3+8\sec t$, and Theresa can paint at a rate of $T(t)=4\cosh t$, where each rate is measured in square feet per hour. Write, but do not evaluate, an integral expression representing the number of square feet they can paint in an 8 hour period.

Problem 1.77 A famed scientist is studying the population growth of a newly-discovered alien microbial species on a newly-discovered planet. She is able to determine a function that appears to model the birth rate for the week she spends observing, $b(t)=\sec(7t)$, in specimens/hour. Her colleague stumbles upon the function that models the death rate, $d(t)=\frac{4}{\tan t}$ in specimens/hour. If there are 483 specimens of the alien when they are first discovered, how many mm of alien population were there at the end of the first 24 hour period of observation? Write, but do not evaluate, the integral that answers this question.

Problem 1.78 It has been shown that Americans are able to find new employment at a rate of $E(t)$, but lose their jobs at a rate of $L(t)$ during a year long study, both measured in people per month. Write, but do not evaluate, an integral expression representing the number of Americans who find a job in the first six months of this particular study.

Problem 1.79 Despite aggressive campaigns to discourage smoking, new people begin smoking on a daily basis. Fortunately, many medications have recently appeared on the market to help smokers kick the habit. The government has funded a decade-long study to measure the rates at which the smoking population changes. The study finds that the rate of new smokers is given by the function $n(t)$, measured in smokers per year, and the rate of smokers quitting is given by the function $q(t)$, also measured in smokers per year. If there are s_0 smokers at the beginning the study, write, but do not evaluate, an integral expression representing how many smokers are there after 10 years.

Problem 1.80 Write a similar problem of your own. You do not need to write an actual equation, you may simply name the function after your favorite letter. Be sure to be creative, and provide a solution. The last sentence of your problem should read: "Write, but do not evaluate an integral expression..."

1.9 Straight-Line Motion: Integration

Objectives — Straight-Line Motion: Integration.

- Determine when the initial condition is relevant to particle motion.
- Distinguish between distance, displacement, and position.
- Find the distance, displacement, and position of a particle using the velocity.

Vocabulary — Displacement.
Displacement is a vector quantity meaning that it can be negative of positive which tells us the direction. Displacement is the total distance you covered from your starting point to your ending point, it doesn't matter what path you took. So for example, everyday you go to school, you may go out with your friends after, then stop at the store, and then go back home. Despite all your detours, your displacement is 0 because you ended the same place you started. Your displacement from home to school could be 1 mile, and your displacement from school to home is -1 mile, because you've gone the opposite direction. **Displacement is easy, we don't have to do anything special to the integral.**

Vocabulary — Distance.
Distance is a scalar, meaning it is always positive and does not tell us direction the way displacement does. While your total displacement every day is 0 (since you end up back at home), your total distance is 2 miles (assuming it's 1 mile in each direction). **Because we're not concerned with negative or positive, we use an absolute value when we do distance.**

Vocabulary — Position.
Position is just what it sounds like. We want to know where a particle is at some certain point. **If we want to know where a particle is after a while, we have to know where it started, aka the initial condition.**

All three of these fit on the first level of our diagram:

f	$\leftrightarrow$	Distance, Displacement, or Position
f'	$\leftrightarrow$	Velocity
f''	$\leftrightarrow$	Acceleration

So, how do differentiate between distance, displacement, or position? It all depends on how you set up the integral. We'll see these set up in each of the following sections.

1.9.1 Finding Displacement

Displacement is the integral that we don't need to do anything special to. *It wakes up like that. Just wakes up and rolls out the door.* This matches what we were doing in the last section where we were finding how much the amount of something changed over a period of time. So we have our first theorem for this section.

Theorem 1.9.1 — Displacement Integral. Displacement of a particle can be evaluated as:

$$\int_a^b f(x)\,\mathrm{d}x$$

It's time to see some examples.

Example 1.81 A particle travels along the $x-$axis in a trajectory described by the function $v(t) = 3t+2$ which is valid for $4 \leq t \leq 15$. Find the displacement of the particle from $t = 5$ to $t = 10$. ■

Because this is displacement, we can simply integrate and we will have our integral:

$$\int_5^{10} (3t+2)\,\mathrm{d}t$$

$$= \left(\frac{3}{2}t^2 + 2t\right)\Big|_5^{10}$$
$$= \left(\frac{3}{2}(10)^2 + 2(10)\right) - \left(\frac{3}{2}(5)^2 + 2(5)\right)$$
$$= (150+20) - \left(\frac{75}{2} + 10\right)$$
$$= 160 - \frac{75}{2} = \frac{320}{2} - \frac{75}{2} = \frac{245}{2}$$

On the AP test, I would write my final answer as: $\int_5^{10} v(t)\,\mathrm{d}t = \frac{245}{2}$

Example 1.82 A particle traveling along with a velocity given by $v(t) = 2t\sin t^2$ maintains this velocity between $t = \sqrt{\frac{\pi}{2}}$ and $t = \sqrt{\pi}$. Find the total displacement of the particle during this interval. ■

Again, we will begin by setting up and evaluating an integral: $\displaystyle\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} 2t\sin t^2\,\mathrm{d}t$.
We will need to use $u-$substitution to evaluate this integral. If we choose $u = t^2$ we would have:

$$u = t^2 \qquad \mathrm{d}u = 2t\,\mathrm{d}t \qquad u\left(\sqrt{\tfrac{\pi}{2}}\right) = \tfrac{\pi}{2} \qquad u\left(\sqrt{\pi}\right) = \pi$$

$$\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} \left(2t\sin(t^2)\right)\,\mathrm{d}t$$

$$= \int_{\frac{\pi}{2}}^{\pi} (\sin u)\,\mathrm{d}t$$
$$= -\cos u|_{\frac{\pi}{2}}^{\pi}$$
$$= (-\cos\pi) - \left(-\cos\frac{\pi}{2}\right) = -(-1) - (-0) = 1$$

My final answer would be written as: $\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} 2t\sin t^2\,\mathrm{d}t = 1$.

Example 1.83 A particle moves according to the function $v(t) = \ln(5t^3 + 3t^2 - t)$ in the interval $t = 3$ to $t = 10$. Find the displacement of the particle in this interval. ■

Once again, we'll need to set up an integral: $\int_3^{10} \ln(5t^3 + 3t^2 - t)\,dt$. We'll need to use a calculator to evaluate this, so we'll get the answer: $\int_3^{10} \ln(5t^3 + 3t^2 - t)\,dt \approx 50.098$.

1.9.2 Finding Distance

For distance, we said we needed to use absolute value because we only want to take positive motion into consideration. In the next chapter, we'll see how to actually approach these by hand. Today, though, we will do these using a calculator. It should be noted that using the absolute value adds quite a level of complexity to an integral, it is not a trivial change. So we have our second theorem.

Theorem 1.9.2 — Distance Integral. Distance a particle travels can be evaluated as:

$$\int_a^b |f(x)|\,dx$$

Let's see some examples to see how these questions are different from the displacement questions.

Example 1.84 A particle travels according to the function $v(t) = 4t + 3 - 3t^2$ which is valid in the interval $t = 0$ to $t = 15$. Set up, but do not evaluate an integral expression to represent the distance the particle travels in the interval from $t = 5$ to $t = 10$. ■

The keyword here is "distance," so we will need to use absolute value for this integral. Of course, we already see an absolute value in the velocity function, but this will not affect our approach. We will have the integral expression: $\int_5^{10} |4t + 3 - 3t^2|\,dt$. If this looks confusing, we could also write: $\int_5^{10} |v(t)|\,dt$.

Example 1.85 The velocity of a particle traveling along the $x-$axis is governed by the function $v(t) = 3\sin t - 2$ in the interval $0 \leq t \leq \pi$. What is the distance the particle travels during this interval? ■

We will need to use a calculator for this question. We will first set up the integral and then evaluate it. Be sure to ask your teacher how to do absolute value in the calculator if you don't already know: $\int_0^{\pi} |3\sin t - 2|\,dt \approx 2.498$ or 2.499.

Example 1.86 A particle travels along the $x-$axis at the velocity given by $v(t) = \cos t + \sin^2 t$ in the first quadrant. What is the distance the particle travels? ■

In this case, the first quadrant is referring to $t = 0$ to $t = \frac{\pi}{2}$, so we will integrate with those bounds: $\int_0^{\pi} |\cos t + \sin^2 t|\,dt \approx 1.785$.

1.9.3 Finding Position

As we said earlier, if we want to know where something ends up, we need to know where it starts. This is similar to the problems we did in the previous problem where we wanted to know how much of something we ended up with – which means we had to know how much we started with. We now have our final theorem for this section.

Theorem 1.9.3 — Position Integral. Position can be evaluated as:

$$f(t_0) + \int_{t_0}^{t} f(x)\,dx$$

where t_0 is the starting time, $f(t_0)$ is the starting position, and t is the time we are interested in.

Now, for some examples.

Example 1.87 A particle's velocity is given by the function $v(t) = 2t^3 - 1$ where t is measured in seconds and $v(t)$ is measured in meters per second. At $t = 0$, the particle is at $x = 4$. Where is the particle at $t = 1$? ■

This is a tricky problem because you are used to x being the independent variable and y being the dependent variable – we usually write our functions as $f(x)$. When we are talking about functions of particle position, however, the function is written as $x(t)$. This means that t is the independent variable and x is the dependent variable. Thinking of this another way, normally x goes along the horizontal axis. On these problems t will instead be on the horizontal axis, and x is on the vertical axis.

So, if we use our approach from Theorem 1.8.1, we get: initial position $+ \int_{\text{initial time}}^{\text{desired time}} v(t)\mathrm{d}t$. We'll integrate this by-hand again:

$$
\begin{aligned}
4 + \int_0^1 (2t^3 - 1)\,\mathrm{d}t \\
&= 4 + \left(\frac{1}{2}t^4 - t\right)\Big|_0^1 \\
&= 4 + \left(\frac{1}{2}\cdot 1^4 - 1\right) - \left(\frac{1}{2}\cdot 0^4 - 0\right) \\
&= 4 + \left(-\frac{1}{2}\right) - (0) = 4 - \frac{1}{2} = \frac{7}{2}.
\end{aligned}
$$

My answer would be written as: $4 + \int_0^1 (2t^3 - 1)\mathrm{d}t = \frac{7}{2}$

Example 1.88 A particle traveling along the $x-$axis is located at $x = 14$ at $t = 12$. If the velocity of the particle is given by $v(t) = \ln\left(t^3\right)$ in the interval $0 \le t \le 30$, where is the particle at $t = 30$? ■

Even though the interval for this function is from $t = 0$ to $t = 30$, we're only going to look at what the particle is doing from $t = 12$ to $t = 30$ since we already know where the particle is at $t = 12$. (We don't need to worry about what happens before $t = 12$, that is irrelevant. Since $u-$substitution would not suffice in this problem, we will need to use the calculator to integrate. We will get our answer of: $14 + \int_{12}^{30} \ln\left(t^3\right)\mathrm{d}t \approx 176.651$.

Example 1.89 The acceleration of a particle is given by the function $a(t) = 4t^3 - 3t$ where t is measured in seconds and $a(t)$ is measured in m/sec^2. If the particle has a velocity of $13m/sec$ at time $t = 1$, what is the velocity at time $t = 2$? ■

This problem is a little different because they have given us the acceleration and asked for the velocity. However, we are still only going through one level of integration (from a to v), so this is actually the same as the other problems we have done.

We want to know what a velocity is at a certain point – in order to know that, we must know what it was to begin with. Consider: If you know how much you sped up by, you still don't know how fast you are going unless you knew how fast you were going to begin with.

We will use the form given in Theorem 1.8.1: initial velocity $+\int_{\text{initial time}}^{\text{desired time}} a(t)\,\mathrm{d}t$. Writing this with the proper numbers input, we have: $13+\int_1^2(4t^3-3t)\,\mathrm{d}t$. This could be the answer you leave for the free response part of the test.

An integral like this would be very likely to appear on the non-calculator section of the test because it is so straight-forward. Let's integrate by-hand.

$$\begin{aligned}
&13+\int_1^2(4t^3-3t)\,\mathrm{d}t\\
&\qquad= 13+\left(t^4-\frac{3}{2}t^2\right)\Big|_1^2\\
&\qquad= 13+\left(2^4-\frac{3}{2}\cdot 2^2\right)-\left(1^4-\frac{3}{2}\cdot 1^2\right)\\
&\qquad= 13+(16-6)-\left(1-\frac{3}{2}\right)\\
&\qquad= 13+10-\left(-\frac{1}{2}\right)=13+10+\frac{1}{2}=23.5
\end{aligned}$$

For my final answer on the AP test, I would do most of that off to the side and then write my final answer as: $13+\int_1^2(4t^3-3t)\,\mathrm{d}t=23.5$.

AP Be very careful. All of these examples have involved integration because we're going from velocity to position (or acceleration to velocity). They make ask you to go from position to velocity (or velocity to acceleration) in which case you would take the derivative.

1.9.4 Practice Problems

1. A particle travels along the $x-$axis with a velocity given by $v(t)=4t^2-2t$ for all time $t\geq 1$. If the particle is at $x=3$ at time $t=1$, what is the position of the particle at $t=4$? *Simplify your answer to an integer, no calculator.*
2. A particle moves along the $x-$axis with a velocity given by $v(t)=8t^2-4$ for time $t\geq 0$. What is the acceleration of this particle at time $t=2$? *Simplify your answer to an integer, no calculator.*
3. A particle travels along a straight line and has a position $x(t)$ at a given time t. If $x(4)=3$ and the velocity of the particle is $v(t)=\frac{1}{t^2}$, what is the position of the particle when $t=6$? *Simplify your answer to an integer, no calculator.*
4. A particle is moving along the $x-$axis. The velocity of the particle at time t is given by the function $v(t)=3t+t^3$. What is the total distance traveled by the particle from time $t=2$ to time $t=10$? *Round your answer to the nearest integer, use a calculator.*
5. A particle moves in straight line with a velocity given by $v(t)=4t+3t^2$ for $t\geq 0$ what is the displacement of this particle from $t=0$ to $t=5$? *Round your answer to the nearest integer, no calculator.*
6. Write, but do not an integral expression for the distance a particle travels if its velocity is given by the function $v(t)=\sin 2t^3+4t$ on the interval $t=3$ to $t=15$. *Round your answer to the nearest integer, no calculator.*
7. A particle moves along a straight line with an acceleration given by $a(t)=4t+2$. At $t=0$, the velocity of the particle is $v=5$. Write, but do not evaluate an integral expression for the distance the particle travels from $t=0$ to $t=10$. *Round to the nearest integer, no calculator.*

8. The acceleration of a particle traveling along the $x-$axis is given by $a(t) = \sin\left(-3t + \frac{5\pi}{2}\right)$ and at $t = \frac{\pi}{3}$, the velocity is $v = 2$. What is the displacement of the particle over the interval $t = 0$ to $t = \pi$? *Round to the nearest integer, no calculator.*
9. If a particle velocity is given by the function $v(t) = \frac{2}{7t}$ and $x(1) = -3$, where is the particle at $t = 4$? *Write your answer as an expression, no calculator.*
10. A particle's velocity can be determined by the function $v(t) = \frac{1}{3t-1}$. What is the displacement of the particle between $t = 2$ and $t = 6$? *Write your answer as an expression, no calculator.*

1.9.5 Practice Problems Solutions

1. 72
2. 32
3. $\frac{37}{12}$
4. 2640
5. 175
6. $\int_3^{15} \left|\sin 2t^3 + 4t\right| \, dt$
7. $\int_0^{10} \left|2t^2 + 2t + 5\right| \, dt$
8. $-\frac{1}{9}\sin\frac{-\pi}{2} + 2\pi + \frac{1}{9}\sin\frac{5\pi}{2} = \frac{2}{9} + 2\pi$
9. $-3 + \frac{2}{7}\ln(4)$
10. $\frac{1}{3}\ln\left(\frac{5}{17}\right)$

1.9.6 Homework: Straight-Line Motion

Problem 1.81 If a particle travels at a velocity of $v(t) = 2t + 3$ and is located at $x = 4$ at $t = 2$, where is the particle at $t = 5$? *Simplify your answer to an integer, no calculator.*

Problem 1.82 A particle is traveling at a velocity given by $v(t) = cos(\frac{\pi}{4}t)$. What is the particle's displacement in the first 6 seconds of traveling? *Simplify your answer to an expression in terms of π, no calculator.*

Problem 1.83 A particle travels along the $x-$axis at a velocity given by $v(t) = 4t(3t-1)$. If $x(0) = 3$, what is $x(2)$? *Simplify your answer to an integer, no calculator.*

Problem 1.84 A particle is at $x = 2$ at $t = 10$. Its velocity is given by $v(t) = e^{4t-1}$. Write, but do not evaluate, an integral expression representing the distance the particle travels from $t = 10$ to $t = 20$.

Problem 1.85 If a function travels at a velocity of $v(t) = (3t^2 - 1)(t^3 - t + 12)$, what is the displacement of the particle in the interval $t = 2$ to $t = 5$? *Simplify your answer to an integer, use a calculator.*

Problem 1.86 Determine the distance a particle travels from $t = 2.3$ to $t = 4$ if the velocity at any given point in that interval is $v(t) = \sin(t^2 - 4t + 1)$. *Round your answer to the nearest thousandth place, use a calculator.*

Problem 1.87 The velocity of a particle is given by the function $v(t) = \sin(4t) + 2t$. What is the displacement of the particle on the interval $0 \le t \le 4$? *Write your answer as an expression, no calculator.*

Problem 1.88 A particle travels along the $x-$axis at a velocity given by $v(t) = \sec(4t)\tan(4t)$ for the interval $t = 2$ to $t = 2.5$. If the particle is at $x = 4$ when $t = 2$, where is the particle at $t = 2.5$? *Write your answer as an expression, no calculator.*

Problem 1.89 The acceleration of a particle traveling along a straight line is given by the function $a(t) = 4t^2 - 3$. If the velocity of the particle is 4 at $t = 0$, what is the displacement of the particle on the interval $0 \le t \le 2$? *Simplify your answer to a rational number, no calculator.*

1.10 Average Value

Objectives — Average Value.

- Explain the difference between the average value and the value.
- Use integration to determine the average value of a function.
- Distinguish between distance, displacement, position, and average value.

Vocabulary — Average Value.
The average value of a function is exactly what it sounds like it should be. For example, the average velocity of a function would be the average of all the different velocities throughout the trajectory of the particle.

1.10.1 Given f and finding the average value of f'

There is some confusion that students experience regarding why we would use integration to compute the average value. The reasoning lies in how the question is phrased for us. Let's see some examples.

Example 1.90 A car starts at $x = 50$ miles and after 3 hours, it has traveled to $x = 125$ miles. What is the average velocity of the car ■

This example is fairly straightforward. In this time interval, it traveled a total of 75 miles. Since it was a 3 hour time period, the average velocity was 25 miles per hour. We can see this in a more formal context:

$$\begin{aligned} v_{avg} &= \frac{125 \text{ miles} - 50 \text{ miles}}{3 \text{ hours}} \\ &= \frac{75 \text{ miles}}{3 \text{ hours}} \\ &= 25 \text{ miles / hour} \end{aligned}$$

Note that in this example, we went from $x(t) \longrightarrow v(t)$. This is an important detail.

Example 1.91 The position of a particle in the interval $0 \le t \le 10$ is given by the function $x(t) = 4t^2 + 2t - 1$, where t is measured in seconds and x is measured in meters. What is the average velocity of this function on this interval? ■

Once again, we will be starting with $x(t)$ and going towards $v(t)$. Recall in example 1.90 we had two $x-$values, specifically $x = 50$ (the starting displacement) and $x = 125$. We then divided those by time. We should do the same here and obtain two $x-$values to start. Using our boundary information:

$$\begin{aligned} x(0) &= 4(0)^2 + 2(0) - 1 = -1 \\ x(10) &= 4(10)^2 + 2(10) - 1 = 419 \end{aligned}$$

We can now do this in the same formal sense that we did the last problem:

$$\begin{aligned} v_{avg} &= \frac{419 \text{ meters} - -1 \text{ meters}}{10 \text{ seconds}} \\ &= \frac{420 \text{ meters}}{10 \text{ seconds}} \\ &= 42 \text{ meters / second} \end{aligned}$$

So, we arrive out our final conclusion that the average velocity is 42 meters per second.

Example 1.92 The population of an alien ant farm at any given point in time could be given by the function $P(t) = \sin\frac{t\pi}{2}$ where t is measured in years in the closed interval $[0,500]$ and P is measured in thousands of alien ants. What is the average rate of change of the population for the interval $[3,6]$? ■

Once again, we're looking at a scenario where we're given the function and then asked about its derivative. So, we will follow the same procedure we did in the previous two problems. We should begin by getting our function values:

$$P(3) = \sin\left(\frac{3\pi}{2}\right) = 1$$
$$P(6) = \sin\left(\frac{6\pi}{2}\right) = 0$$

Now, we can simply plug in as we did before:

$$v_{avg} = \frac{0-1}{6-3} \quad = \frac{-1}{3}$$

We conclude with the answer that the average rate of change for $t = 3$ to $t = 6$ is $-\frac{1}{3}$ thousands of aliens per year.

Notice that when we do these problems, we are simply using the slope formula. We might formalize this and say that:

Theorem 1.10.1 When given f and finding the average value of f', we are simply finding the slope of the secant line, therefore, we should use the slope formula:

$$m = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

1.10.2 Given f' and finding the average value of f'

In the previous section, we discussed the scenario when we would be given an f and then be asked to find the average value of f'. What happens if we are given f' and asked to find the average value of f'?

The underlying math is exactly the same – we will still be using the slope formula and finding the slope of the secant line. However, in order to get the values for our numerator, we must integrate.

Example 1.93 Let the velocity of a particle be given by the function $v(t) = 4t^2 + 2t - 1$ on the interval $[0,10]$ where t is measured in seconds and v is measured in meters per second. What is the average velocity in this interval? ■

So notice that we are not given x at all. We begin with v and we should end with v. However, since we are still using the same approach as we did in the earlier problems, we will need to find a $y_2 - y_1$ for the numerator. i.e. we have:

$$v_{avg} = \frac{???}{10-0}$$

In order to get the numerator, we will use integration. When you consider the way the FTC works, you can see that:

$$\int_0^{10} v(t)\,\mathrm{d}t = x(10) - x(0)$$

Consider that when we perform that integral, we will find the antiderivatives (which are the position functions) and then plug in the bounds (which gives us the $y-$values that we were using in the previous examples. Therefore, we have:

$$v_{avg} = \frac{\int_0^{10} v(t)\,\mathrm{d}t}{10-0} = \frac{\int_0^{10}(4t^2+2t-1)\,\mathrm{d}t}{10}$$

This integral seems a little complicated. Most people would simply write these integrals like this instead:

$$v_{avg} = \frac{1}{10}\int_0^{10}(4t^2+2t-1)\,\mathrm{d}t$$

We can now just integrate this as we normally would:

$$\begin{aligned} v_{avg} &= \frac{1}{10}\int_0^{10}(4t^2+2t-1)\,\mathrm{d}t \\ &= \frac{1}{10}\left(\frac{4}{3}t^3+t^2-t\right)\Bigg|_1^{10} \\ &= \frac{1}{10}\left[\left(\frac{4}{3}(10)^3+(10)^2-(10)\right)-\left(\frac{4}{3}(0)^3+(0)^2-(0)\right)\right] \end{aligned}$$

As far as the free response part of the AP test is concerned, there's no reason to simplify beyond that. However, we can continue to simplify since we're not taking the test right at this moment.

$$\begin{aligned} &= \frac{1}{10}\left[\left(\frac{4}{3}(10)^3+(10)^2-(10)\right)-\left(\frac{4}{3}(0)^3+(0)^2-(0)\right)\right] \\ &= \frac{1}{10}\left(\frac{4\cdot 1000}{3}+100-10\right) \\ &= \frac{1}{10}\left(\frac{4000}{3}+\frac{300}{3}-\frac{30}{3}\right) \\ &= \frac{1}{10}\left(\frac{4000+300-30}{3}\right) \\ &= \frac{1}{10}\left(\frac{4270}{3}\right) \end{aligned}$$

This is our final answer in proper AP formatting and we should always remember to include units: $\frac{1}{10}\int_0^{10} v(t)\,\mathrm{d}t = \frac{427}{3}$ meters per second
We now have a general form that we can use when we're given f' and then asked to find the average value of f':

Theorem 1.10.2 The average value of f' on the interval $[a,b]$ can be calculated using the formula:

$$f'_{avg} = \frac{1}{b-a}\int_a^b f'(t)\,\mathrm{d}t$$

Example 1.94 Water leaks from a faucet at a rate of $L(t) = \sin(3t)$ mL per minute during $[0,60]$. What is the average rate of water leaking in the interval $[4,8]$? ■

Since we're starting at the first derivative and the question is asking us about the first derivative, we will plug directly into Theorem 1.10.2 and then solve. Note that we will be using $u-$substitution to find the antiderivative.

$$\begin{aligned}
L_{avg} &= \frac{1}{8-4}\int_4^8 L(t)\,\mathrm{d}t \\
&= \frac{1}{4}\int_4^8 \sin(3t)\,\mathrm{d}t \\
&= \frac{1}{4}\left(-\frac{1}{3}\cos(3t)\right)\Big|_4^8 \\
&= \frac{1}{4}\left(-\frac{1}{3}\cos(24) + \frac{1}{3}\cos(12)\right) \\
&= \frac{1}{12}(-\cos(24) + \cos(12))
\end{aligned}$$

Simplifying beyond this would be fruitless as $\cos(24)$ and $\cos(12)$ are both irrational. Therefore, we have our final answer as we would write it on the AP test (including units!):
$\frac{1}{4}\int_4^8 L(t)\,\mathrm{d}t = \frac{1}{12}(-\cos(24) + \cos(12))$ mL per minute.

1.10.3 Monkey Wrenches

One of the most common problems students have is in dealing with integrals that have been split up and/or are abstract. These problems are not so hard if we simply allow ourselves to become comfortable with them.

Example 1.95 Consider a function $f(t)$ such that $\int_0^6 f(t)\,\mathrm{d}t = 4$ and $\int_9^6 f(t)\,\mathrm{d}t = 12$. What is the average value of f on the interval $[0,9]$? ■

We are beginning with f and finding the average value of f. Therefore, we should use the formula we saw in Theorem 1.10.2:

$$f_{avg} = \frac{1}{9}\int_0^9 f(t)\,\mathrm{d}t$$

Because of the structure of the information we were given, we should split this up into two integrals:

$$f_{avg} = \frac{1}{9}\left(\int_0^6 f(t)\,\mathrm{d}t + \int_6^9 f(t)\,\mathrm{d}t\right)$$

At this point, we will simply plug in. We know that the first integral, $\int_0^6 f(t)\,\mathrm{d}t = 4$. The second integral will have to be flipped because the bounds were given to us in the wrong order. Therefore, the sign of our answer will change: $\int_6^9 f(t)\,\mathrm{d}t = -12$. We would now just plug these into the equation we developed earlier.

$$
\begin{aligned}
f_{avg} &= \frac{1}{9}\left(\int_0^6 f(t)\,\mathrm{d}t + \int_6^9 f(t)\,\mathrm{d}t\right) \\
&= \frac{1}{9}(4+-12) \\
&= \frac{1}{9}(-8) \\
&= -\frac{8}{9}
\end{aligned}
$$

Since we were not given units, we would not worry about them. Our final answer could be written as: $\frac{1}{9}\int_0^9 f(t)\,\mathrm{d}t = -\frac{8}{9}$

Example 1.96 Let f be the function defined by

$$
f(x) = \begin{cases} \sin(x\pi), & x < 0 \\ 2x, & 0 \le x \le 3 \end{cases}
$$

Find the average value of f on the interval $[-1,3]$. ■

This will work the same way Example 1.95 did. We will need to break up the integral at $x = 0$ since that's where the piecewise-defined function is broken. Notice that we would use $u-$substitution to find the antiderivative of the first part of the function.

$$
\begin{aligned}
f_{avg} &= \frac{1}{3--1}\int_{-1}^{3} f(x)\,\mathrm{d}x \\
&= \frac{1}{4}\left(\int_{-1}^{0} f(x)\,\mathrm{d}x + \int_0^3 f(x)\,\mathrm{d}x\right) \\
&= \frac{1}{4}\left(\int_{-1}^{0} \sin(x\pi)\,\mathrm{d}x + \int_0^3 2x\,\mathrm{d}x\right) \\
&= \frac{1}{4}\left(-\frac{1}{\pi}\cos(x\pi)\Big|_{-1}^{0} + x^2\Big|_0^3\right) \\
&= \frac{1}{4}\left(-\frac{1}{\pi}[\cos(0\cdot\pi) - \cos(-1\cdot\pi)] + [3^2 - 0^2]\right)
\end{aligned}
$$

We could stop here for the AP test. For the sake of this book, however, let's continue.

$$
\begin{aligned}
&= \frac{1}{4}\left(-\frac{1}{\pi}[\cos(0\cdot\pi) - \cos(-1\cdot\pi)] + [3^2 - 0^2]\right) \\
&= \frac{1}{4}\left(-\frac{1}{\pi}[\cos(0) - \cos(-\pi)] + 9\right) \\
&= \frac{1}{4}\left(-\frac{1}{\pi}[1+1] + 9\right) \\
&= \frac{1}{4}\left(-\frac{1}{\pi}\cdot 2 + 9\right) \\
&= \frac{1}{4}\left(-\frac{2}{\pi} + \frac{9\pi}{\pi}\right) \\
&= \frac{1}{4}\left(\frac{9\pi - 2}{\pi}\right) \\
&= \frac{9\pi - 2}{4\pi}
\end{aligned}
$$

We have finally arrived at our answer: $\frac{1}{4}\int_{-1}^{3} f(x)\,dx = \frac{9\pi-2}{4\pi}$.

1.10.4 Homework: Average Value

Problem 1.90 Find the average value of the function $f(x) = \sqrt{3x+2}$ on the closed interval $[1,4]$. Do not use a calculator.

Problem 1.91 Let f be a function such that $\int_{5}^{10} f(x)\,dx = 3$ and $\int_{5}^{0} f(x)\,dx = -2$. What is the average value of f on the closed interval $[0,10]$? Do not use a calculator.

Problem 1.92 A brave person has bungee jumped off of a bridge. For a given time $0 \leq t \leq 9.5$, their height as a function of time is given by the function $f(t) = (t-9.4)\sin^2(t-9.4)+7.9$. What is their average height from $t=2$ to $t=5$? Use a calculator to evaluate.

Problem 1.93 A traveling car has a position given by the function $x(t) = 8t-4$ in the interval $0 \leq t \leq 10$ where t is measured in hours and x is measured in miles. Find the average velocity of the car from $t=2$ to $t=5$.

Problem 1.94 A population of aardvarks has invaded El Paso and scientists are stumped. Scrambling to find answers, they have determined a set of functions that models their growth. The aardvarks are being born at a rate of $B(t) = 400e^{.02t}$ where $B(t)$ is measured in hundreds of aardvarks and t is measured in weeks. They are dying at a rate of $D(t) = 100e^{.01t}$ where $D(t)$ is also measured in hundreds of aardvarks. Answer each of the following using the appropriate units. Use a calculator to evaluate.

a) What is the average number of births in the interval $t=1$ to $t=3$?
b) What is the average number of deaths in the interval $t=1$ to $t=3$?
c) What is the average rate of change of the aardvark population in the interval $t=1$ to $t=3$?

Problem 1.95 Let $x(t) = \sin(3t^2+2t)$ be the position of a particle at a given time t. What is the instantaneous velocity of the particle at $t=3$? Use a calculator to evaluate.

Problem 1.96 A spacecraft travels at a velocity given by the function $v(t) = t\cdot\cos(t+3)$ where $0 \leq t \leq 50$ is measured in seconds and $v(t)$ is measured in meters per second. At time $t=0$, the spacecraft is at $x=400$. Answer each of the following using appropriate units. Use a calculator to evaluate.

a) What is the position of the spacecraft at time $t=50$?
b) What is the distance traveled by the spacecraft in the first 50 seconds of its flight?
c) What is the displacement of the spacecraft in the first 50 seconds of its flight?
d) What is the average velocity of the spacecraft in the first 50 seconds of its flight?
e) What is the average acceleration of the spacecraft in the first 50 seconds of its flight?

Problem 1.97 Let f be the function defined by

$$f(x) = \begin{cases} 2x^2-3x+1, & x<5 \\ 4x^3, & 5 \leq x \leq 10 \end{cases}$$

What is the average value of $f(x)$ in the closed interval $[1,10]$? Do not use a calculator.

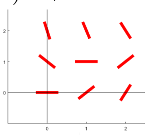
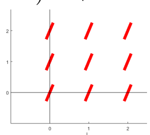
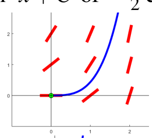
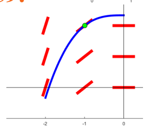
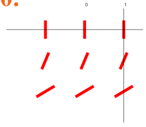
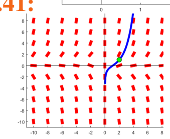
Problem 1.98 Let $v(t) = 2t^2-4$ represent the velocity of a particle in meters per second as a function of seconds on the closed interval $0 \leq t \leq 4$. Find the average velocity of this particle on that interval. Use the appropriate units. Do not use a calculator.

Problem 1.99 Let $x(t) = 2t^2-4$ represent the position of a particle in meters as a function of seconds on the closed interval $0 \leq t \leq 4$. Find the average velocity of this particle on that interval. Use the appropriate units. Do not use a calculator.

Problem 1.100 Consider a function f such that $\int_1^6 f(x)\,dx = 12$ and $\int_3^6 f(x)\,dx = -3$. What is the average value of f in the closed interval $1 \le x \le 3$? Do not use a calculator.

Problem 1.101 Find the average value of the function $f(x) = 3x^2 + 3$ on the closed interval $[-5, -1]$.

1.11 Homework Solutions

Problem 1.1: $4x + C$ **Problem 1.2:** $x^2 + C$ **Problem 1.3:** $ex + C$ **Problem 1.4:** $m^3 - m^2 + \frac{3}{2}m^4 - 4m + C$ **Problem 1.5:** $\frac{1}{5}x^{\frac{5}{2}} + \frac{6}{5}x^{\frac{10}{3}} + C$ **Problem 1.6:** $\frac{x^{e\pi+1}}{e\pi+1} + C$ **Problem 1.7:** $\frac{16}{3}y^3 + 8y^2 + 4y + C$ **Problem 1.8:** $\frac{6}{5}x^{\frac{5}{3}} + \frac{3}{2}x^{\frac{8}{3}} + C$ **Problem 1.9:** $-4\frac{1}{x} - \frac{1}{x^2} + \frac{1}{4x^4} + C$ **Problem 1.10:** $xy + C$ **Problem 1.11:** $\frac{3}{2}\ln|x| + \frac{1}{2}e^{2x} + C$ **Problem 1.12:** $2x^2 - e^{x+4} + C$ **Problem 1.13:** $\frac{3}{2}e^{2x} + C$ **Problem 1.14:** $4\ln|x| - 4x^{\frac{1}{2}} + C$ **Problem 1.15:** $\frac{1}{3}e^{3x} + 2x + C$ **Problem 1.16:** $3e^{3x} + C$ **Problem 1.17:** $4\ln|x| - \frac{2}{x} + \frac{1}{3x^3} + C$ **Problem 1.18:** $-\frac{4}{3}\cos(3x) + C$ **Problem 1.19:** $\frac{1}{2}\tan(2x) + C$ **Problem 1.20:** $\frac{1}{6}x^2 + \csc x + C$ **Problem 1.21:** $x + \tan x + C$ **Problem 1.22:** $2x^2 + x^3 + \sin x + C$ **Problem 1.23:** $4\tan x + 2\ln|x| + \frac{3}{x} + \sin x + C$ **Problem 1.24:** $y\cos x + C$ **Problem 1.25:** $-2\cos\sqrt{x} + C$ **Problem 1.26:** $\frac{1}{6}\tan(6x-5) + C$ **Problem 1.27:** $-\cos(\ln x) + C$ **Problem 1.28:** We cannot solve this with the tools we have. **Problem 1.29:** $\frac{1}{3}\ln|3x-4| + \frac{1}{2}x^6 - x^2 + 5x + C$ **Problem 1.30:** $\frac{9}{5}x^5 - 2x^3 + x + C$ **Problem 1.31:** $-3\cos(x^2 + 3x) + C$ **Problem 1.32:** $\frac{1}{3}(3x^2+5)^3 + C$ **Problem 1.33:** $\frac{9}{5}x^5 + 10x^3 + 25x + C$ **Problem 1.34:** $\frac{1}{4}e^{4\sin(3x-1)} + C$ **Problem 1.35:** $e^{x^2-10} + C$ **Problem 1.36:** $\frac{1}{11}(2x^2-5x)^{11} + C$ **Problem 1.37:** $-\frac{1}{2}(e^x-5)^{-2} + C$ **Problem 1.38:** $\frac{1}{2}\sin^2 x + C$ or $-\frac{1}{2}\cos^2 x + C$ **Problem 1.39:**  **Problem 1.40:**  **Problem 1.41:**  **Problem 1.42:**  **Problem 1.43:**  **Problem 1.44:**  **Problem 1.45:** $y = e^{x^2} - 1 - e^4$ **Problem 1.46:** $y = 2e^x + 2x + 1$ **Problem 1.47:** $y = -\cot x + 2$ **Problem 1.48:** $y = \frac{3}{2}x^2 - 4x + 7$ **Problem 1.49:** $y = t^3 + t - 4$ **Problem 1.50:** $y = \ln|t| + t^2 + 3$ **Problem 1.51:** $y = 2(4x+2)^3 - 4$ **Problem 1.52:** $y = \frac{1}{4}e^{4x} - 2 - \frac{1}{4}e$ **Problem 1.53:** $J = e^m e^{-2}$ **Problem 1.54:** $L = \frac{e^t}{e} - 6$ **Problem 1.55:** $y = \cos^{-1}(-\sin(x))$ **Problem 1.56:** $y = \sqrt{x^2+16}$ **Problem 1.57:** $x = \left(\frac{3}{8}y^2 - \frac{3}{2}\right)^{\frac{2}{3}}$ **Problem 1.58:** $y = e^{-24}e^{3x}$ **Problem 1.59:** $\frac{\sqrt{2}}{2}$ **Problem 1.60:** $-\frac{1}{12}$ **Problem 1.61:** 28 **Problem 1.62:** -3 **Problem 1.63:** $\frac{1}{27}(7^9 - 1) + \frac{2}{3}$ **Problem 1.64:** 4 **Problem 1.65:** $\frac{2}{3} + \frac{1}{2}\pi^2$ **Problem 1.66:** $\frac{5^6-19}{6} + 375$ **Problem 1.67:** $2a$ **Problem 1.68:** $2(f^2 - b^2) + e^f - e^b$ **Problem 1.69:** -7 **Problem 1.70:** $\frac{1}{2}(e - e^9)$ **Problem 1.71:** b **Problem 1.72:** a **Problem 1.73:** 58 bushels **Problem 1.74:** 912 hamburgers **Problem 1.75:** 401.920 pounds **Problem 1.76:** $\int_0^8 (D(t) + T(t))\,dt$ **Problem 1.77:** $483 + \int_0^{24}(b(t) - d(t))\,dt$ **Problem 1.78:** $\int_0^6 E(t)\,dt$ **Problem 1.79:** $s_0 + \int_0^{10}(n(t) - q(t))\,dt$ **Problem 1.80:** Answers will be graded based on creativity and effort. **Problem 1.81:** 34 **Problem 1.82:** $\frac{-4}{\pi}$ **Problem 1.83:** 27 **Problem 1.84:** $\int_{10}^{20} \left|e^{4t-1}\right| dt$ **Problem 1.85:** 8550 **Problem 1.86:** 1.051 **Problem 1.87:** $\frac{65}{4} - \frac{\cos(16)}{4}$ **Problem 1.88:** $4 + \frac{1}{4}(\sec 10 - \sec 8)$ **Problem 1.89:** $\frac{22}{3}$ **Problem 1.90:** $\frac{2}{27}\left(14\sqrt{(14)} - 5\sqrt{5}\right)$ **Problem 1.91:** $\frac{1}{2}$ **Problem 1.92:** 5.229 **Problem 1.93:** 8 mph **Problem 1.94:** a) 416 births per week, b) 102 deaths per week, c) the population is growing by 314 aardvarks per week during this time interval. **Problem 1.95:** -0.2655 **Problem 1.96:** a) 419.868 b) 788.911, c) 19.868, d) 0.397, e) -0.918 **Problem 1.97:** $\frac{1}{9}\left[\frac{2}{3}(5)^3 - \frac{3}{2}(5)^2 + 1 - \frac{2}{3} + \frac{3}{2} - 1 + 10^4 - 5^4\right]$ **Problem 1.98:** $\frac{20}{3}$ meters per second **Problem 1.99:** 8 meters per second **Problem 1.100:** $\frac{15}{2}$ **Problem 1.101:** 34

1.12 Enrichment: Using Partial Fractions

1.13 Enrichment: Integration by Parts

1.14 Nerd Night Review 7

Can you do these without mixing up integrals and derivatives? Have your teacher organize a Nerd Night at school, or get some friends together and have your own Nerd Night at home.

Evaluate each of the following:

1. $\frac{d}{dx}\left(4x^2 \sin x\right) =$
2. $\frac{d}{dx}\left(2x^2-5\right)(3x+7) =$
3. $\int x \sin\left(3x^2\right) \mathrm{d}x =$
4. $\frac{d}{dx} \tan 2x^8 =$
5. $\int \frac{4}{8x-15} \mathrm{d}x =$
6. $\int e^{8x+15} \mathrm{d}x =$
7. $\int \sec^2 x \tan x \, \mathrm{d}x =$
8. $\int 4x(3x^3+2x) \mathrm{d}x =$
9. $\frac{d}{dx} 4x(3x^3+2x) =$
10. $\frac{d}{dx} \frac{5x-3}{2x^2+5} =$
11. $\frac{d}{dx} \frac{5x^8+3x}{2} =$
12. $\int \frac{3}{e^{2x}} \mathrm{d}x =$
13. $\int_0^{10} \sin \frac{\pi\theta}{5} \mathrm{d}\theta =$
14. $\frac{d}{dx} \sec\left(4x^2-3x+1\right) =$
15. $\int_0^1 (8x-5)(4x^2-5x+2)^8 \mathrm{d}x =$
16. $\int_1^3 \frac{1}{x^2} \mathrm{d}x =$
17. $\frac{d}{dx} \frac{1}{x^2} =$
18. $\int_{-2}^4 \frac{1}{4x+3} \mathrm{d}x =$
19. $\int_1^5 \frac{1}{2x-15} \mathrm{d}x =$
20. $\frac{d}{dx}(14x-2+5\ln 2x) =$

Answers

1. $4x^2 \cos x + 8x \sin x$
2. $18x^2+28x-15$
3. $-\frac{1}{6}\cos 3x^2 + C$
4. $16x^7 \sec^2 2x^8$
5. $\frac{1}{2}\ln|8x-15|+C$
6. $\frac{1}{8}\left(e^{8x+15}\right)+C$
7. $\frac{1}{2}\sec^2 x + C$ or $\frac{1}{2}\tan^2 x + C$

   (Both are possible answers)
8. $\frac{12}{5}x^5+\frac{8}{3}x^3+C$
9. $16x\left(3x^2+1\right)$
10. $\frac{-10x^2-12x-25}{(2x^2+5)^2}$
11. $\frac{1}{2}\left(40x^7+3\right)$
12. $-\frac{3}{2}e^{-2x}+C$
13. 0
14. $(8x-3)\sec\left(4x^2-3x+1\right)\tan\left(4x^2-3x+1\right)$
15. $-\frac{511}{9}$
16. $\frac{2}{3}$
17. $-\frac{2}{x^3}$
18. Undefined. (Look at the graph.)
19. $\frac{1}{2}\ln\left(\frac{5}{13}\right)$
20. $14+\frac{5}{x}$

Index

Symbols

A

B

C

D